ESTIMATING RESIDUAL ERROR BY REPEATED MEASUREMENTS

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(Received March 1992)

ABSTRACT

Repeated measurements can be used to estimate the residual error of a measurement process. Residual error, defined as the error remaining after all known sources of error have been accounted for, is what causes differences in the measurement outcome when everything about the measurement process is seemingly identical. Four estimators for the amount of residual error are suggested. These estimators are functions of the outcomes from m repeated measurements on each of n items. Assuming a normal distribution for the residual error, two of the estimators are unbiased estimators for the standard deviation of the residual error, and the third is the maximum likelihood estimator for the standard deviation of the residual error. The fourth is not an estimator for standard deviation, but rather it uses the distance between measurement order statistics as an indicator of the amount of residual error. The efficiencies of the first two estimators and the bias of the maximum likelihood estimator are evaluated. Computations use standard statistical methods and are included in appendices.

This work, motivated by the study of machines used to measure the modulus of elasticity of dimension lumber, has been used to assess the performance of this machinery. An example using data from more than 10 years ago and some recent data show that the residual error then was about double that of today for well-tuned, high-speed production-line machinery.

Keywords: Repeatability, residual error, repeated measurements.

INTRODUCTION

Error is a part of every measurement. In fact, a measurement reported without an indication of measurement error can be considered incomplete. The terms *accuracy, precision, discrepancy, consistency,* and *repeatability* have all been used in the discussion of errors (Frank 1959).

The component of measurement error under consideration here is the residual error, which is the error remaining after accounting for all known systematic errors. Systematic errors include those due to miscalibration, computational mistakes, effects of environment such as temperature and humidity, damaged equipment, and peculiarities in taking readings. Residual error is the random component of error that causes differences in readings to occur with seemingly identical measurements of the same item. As more is learned about the measurement process and refinements are made, some of what was residual error can become systematic error that can be corrected or compensated.

Residual error may be observed for all measurement processes that have a sufficiently small resolvable unit, i.e., sufficiently high precision. It is possible that there is not enough precision for the residual error to be evident. In that case, one can state only that the residual error is less than the measurement precision.

A quantitative measure of residual error is useful in the error analysis of the measurement. It also allows the measurement process to be compared with other similar processes or with itself before and after various adjustments and improvements are made. For example, in machine stress rating of dimension lumber, the modulus of elasticity of each piece of lumber

Wood and Fiber Science, 25(1), 1993, pp. 46–60

is measured by high speed production-line machinery. Residual error comparisons of production-line measurements from different machines or from the same machine at different speeds or at different times are important because small differences in residual error can lead to significant differences in high grade recovery and hence profitability for the producer. As the machine changes, either through wear, replacement of parts or improvements, the residual error can be tracked and decisions made accordingly.

The object here is to suggest some practical estimators for the residual error of measurement processes. Four alternatives and their properties are discussed. Each of the alternatives has its advantages and disadvantages involving performance and ease of computation. When comparisons are made, the same estimator should be used; otherwise, different applications may call for different estimators.

These estimators or similar ones likely have been used elsewhere as they are based on standard statistical methods. More common than the Rayleigh variates used here, most such analyses use their squares, i.e., their corresponding chi-square variates. Excellent references are Wilks 1962; Miller 1964; Rao 1965; and Hogg and Craig 1965. The results presented here are intended to be in a form that is immediately applicable for assessing the repeatability of machines used in wood products testing. Application to production-line equipment for measuring modulus of elasticity has proved to be valuable in tracking machine performance, finding problems, and improving profitability of the production process (Logan 1991).

Because the estimators are based on repeated measurements of the same quantity, the methods described are not appropriate for measurement processes that are destructive or otherwise significantly alter the measured quantity.

It will be seen that under the assumption of normality for the residual error, two of the suggested estimators are unbiased estimators for the standard deviation of the residual error, and the third is the maximum likelihood estimator for this standard deviation. Although the maximum likelihood estimator is biased, the bias is small for reasonable sample sizes. The fourth estimator is not an estimator for standard deviation of the residual error, but rather it uses the distance between measurement order statistics as an indicator of the amount of residual error.

MEASUREMENT DEFINITION

We assume m repeated measurements on each of n items. If the measurement process is perfectly repeatable with no residual error, then a complete description is given by n values, one for each item. In the usual situation, we obtain the m measured values $z_j(i)$, $j = 1, \ldots, m$ for each item i where the item identifying index i ranges from 1 to n.

The m measurements for each item can be placed in order from smallest to largest. The notation $z_{(j)}(i)$, $j = 1, \ldots, m$ is used for the ordered measurements such that $z_{(1)}(i) \leq z_{(2)}(i) \leq \ldots \leq z_{(m)}(i)$.

ESTIMATOR $S_{(1)}$

The estimator S_1 for the standard deviation of the residual error is defined by:

$$S_1 = \frac{\Gamma((m-1)/2)}{\sqrt{2}n\Gamma(m/2)} \sum_{i=1}^{n} \left(\sum_{j=1}^{m} (z_j(i) - \bar{z}(i))^2 \right)^{1/2} \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function and $\bar{z}(i)$ is the sample mean for the m measurements of item i.

TABLE 1. *Coefficient of variation R_1 for estimator S_1.*

No. of pieces (n)	No. of measurements for each piece (m) 2	3	4	5	10	20
1	0.7555	0.5227	0.4220	0.3630	0.2388	0.1633
2	0.5342	0.3696	0.2984	0.2567	0.1688	0.1154
3	0.4362	0.3108	0.2437	0.2096	0.1379	0.0943
4	0.3778	0.2614	0.2110	0.1815	0.1194	0.0816
5	0.3379	0.2338	0.1887	0.1623	0.1068	0.0730
10	0.2389	0.1653	0.1335	0.1148	0.0755	0.0516
15	0.1951	0.1350	0.1090	0.0937	0.0616	0.0422
20	0.1689	0.1169	0.0944	0.0812	0.0534	0.0365
25	0.1511	0.1045	0.0844	0.0726	0.0478	0.0327
30	0.1379	0.0954	0.0770	0.0663	0.0436	0.0298
35	0.1277	0.0884	0.0713	0.0614	0.0404	0.0276
40	0.1195	0.0826	0.0667	0.0574	0.0378	0.0258
45	0.1126	0.0779	0.0629	0.0541	0.0356	0.0243
50	0.1068	0.0739	0.0597	0.0513	0.0338	0.0231

$$\bar{z}(i) = \frac{1}{m}\sum_{j=1}^{m} z_j(i) \tag{2}$$

An important question concerning the estimator S_1 is how many measurements must be taken to obtain S_1 with prescribed accuracy. We can study this after obtaining the mean and variance of the estimator S_1. Assuming the residual error is normally distributed, the result, derived in Appendix A, is:

$$E[S_1] = \sigma \tag{3}$$

$$Var[S_1] = \frac{\sigma^2}{n}\left(((m-1)/2)\frac{\Gamma^2((m-1)/2)}{\Gamma^2(m/2)} - 1\right) = \sigma^2 R_1{}^2 \tag{4}$$

where σ is the residual error standard deviation. The ratio R_1 given by

$$R_1 = \frac{(Var[S_1])^{1/2}}{E[S_1]} = \frac{1}{\sqrt{n}}\left(((m-1)/2)\frac{\Gamma^2((m-1)/2)}{\Gamma^2(m/2)} - 1\right)^{1/2} \tag{5}$$

is a measure of the variation of S_1 relative to its expected value (the coefficient of variaton of S_1).

The factor $1/\sqrt{n}$ shows how R_1 decreases as n increases and the factor

$$Q_m = \left(((m-1)/2)\frac{\Gamma^2((m-1)/2)}{\Gamma^2(m/2)} - 1\right)^{1/2} \tag{6}$$

shows how R_1 varies with m (decreases as m increases). Table 1 lists R_1 for several values of n and m.

ESTIMATOR S_2

The estimator S_2 defined by:

$$S_2 = \frac{\Gamma(n(m-1)/2)}{\sqrt{2}\Gamma((n(m-1)+1)/2)}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(z_j(i) - \bar{z}(i))^2\right)^{1/2} \tag{7}$$

TABLE 2. *Coefficient of variation R_2 for estimator S_2.*

No. of pieces (n)	No. of measurements for each piece (m) 2	3	4	5	10	20
1	0.7555	0.5227	0.4220	0.3630	0.2388	0.1633
2	0.5227	0.3630	0.2941	0.2536	0.1678	0.1151
3	0.4220	0.2941	0.2388	0.2061	0.1367	0.0939
4	0.3630	0.2536	0.2061	0.1781	0.1183	0.0812
5	0.3232	0.2262	0.1840	0.1591	0.1057	0.0726
10	0.2262	0.1591	0.1296	0.1121	0.0746	0.0513
15	0.1840	0.1296	0.1057	0.0915	0.0609	0.0419
20	0.1591	0.1121	0.0915	0.0792	0.0527	0.0363
25	0.1421	0.1002	0.0818	0.0708	0.0472	0.0325
30	0.1296	0.0915	0.0746	0.0646	0.0431	0.0296
35	0.1199	0.0847	0.0691	0.0598	0.0399	0.0274
40	0.1121	0.0792	0.0646	0.0559	0.0373	0.0257
45	0.1057	0.0746	0.0609	0.0527	0.0351	0.0242
50	0.1002	0.0708	0.0578	0.0500	0.0333	0.0229

is a more efficient estimator for σ than is S_1. For the same n and m values, the ratio

$$R_2 = \frac{(\mathrm{Var}[S_2])^{1/2}}{E[S_2]} \tag{8}$$

is less than or equal to R_1. This means that a smaller number of measurements will give the same estimator accuracy. However, for values m and n of interest, the difference may not be large, and, at least for the case m = 2, the computational simplicity of S_1 can weigh in its favor.

Again, assuming a normal distribution, the expected value and variance of S_2, derived in Appendix B, are:

$$E[S_2] = \sigma \tag{9}$$

$$\mathrm{Var}[S_2] = \sigma^2\left(\frac{n(m-1)}{2}\frac{\Gamma^2(n(m-1)/2)}{\Gamma^2((n(m-1)+1)/2)} - 1\right) = \sigma^2 R_2{}^2 \tag{10}$$

where the ratio R_2 is:

$$R_2 = \left(\frac{n(m-1)}{2}\frac{\Gamma^2(n(m-1)/2)}{\Gamma^2((n(m-1)+1)/2)} - 1\right)^{1/2} \tag{11}$$

Table 2 lists R_2 for several values of n and m. Comparison of R_2 with R_1 shows R_2 is smaller than R_1 and the difference is greatest for small m; however, even for small m, the difference is not severe. For example, if we measure each item twice (m = 2), then measurement of 40 items gives about the same variance for estimator S_1 as does measurement of 35 items for estimator S_2.

EFFICIENCY OF UNBIASED ESTIMATORS

There is a theoretical lower bound for the variance of any unbiased estimator. The efficiency of any particular estimator is defined by the ratio of the lower bound variance to the actual variance. For the present case where we take m measurements of each of n items, the lower bound variance for any unbiased estimator of the residual error standard deviation is computed in Appendix C as:

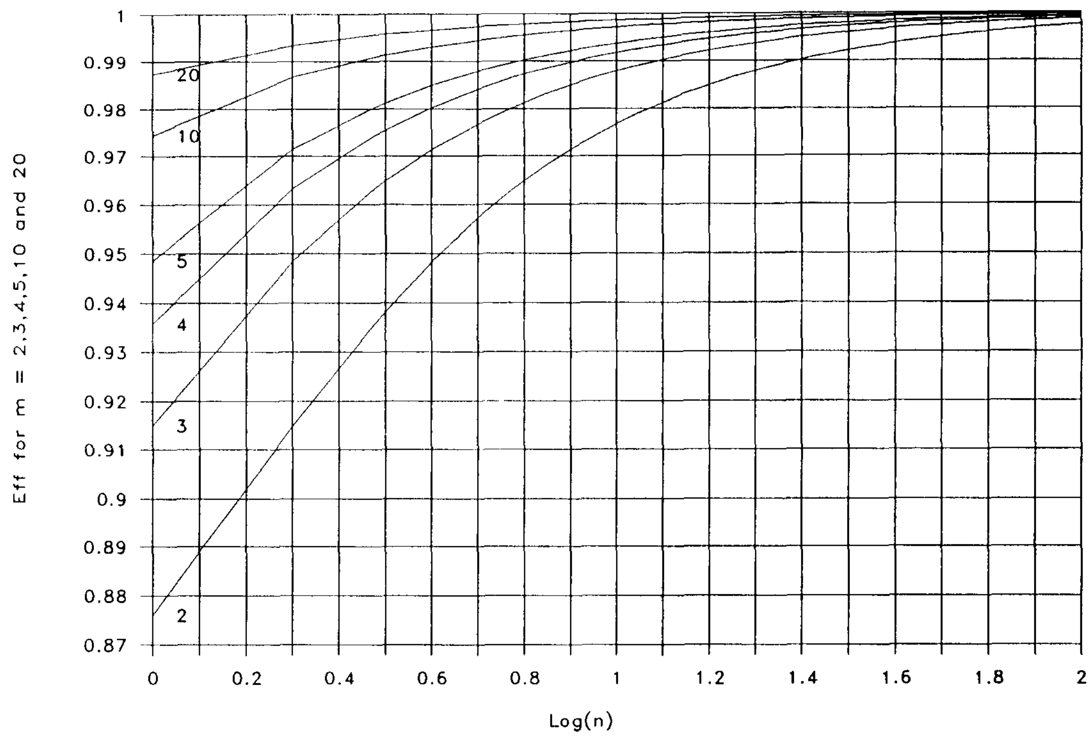

FIG. 1. Efficiency of estimator S_2 versus log(n) for m = 2, 3, 4, 5, 10 and 20.

$$\mathrm{MinVar} = \frac{\sigma^2}{2n(m-1)} \tag{12}$$

Then, the efficiencies of estimators S_1 and S_2 are:

$$\mathrm{Eff}\,S_1 = \left\{2(m-1)\left(\frac{(m-1)}{2}\frac{\Gamma^2((m-1)/2)}{\Gamma^2(m/2)} - 1\right)\right\}^{-1} \tag{13}$$

$$\mathrm{Eff}\,S_2 = \left\{2n(m-1)\left(\frac{n(m-1)}{2}\frac{\Gamma^2(n(m-1)/2)}{\Gamma^2((n(m-1)+1)/2)} - 1\right)\right\}^{-1} \tag{14}$$

It is interesting that the efficiency of S_1 does not change with sample size n. Figure 1 is a graph of Eff S_2 versus log(n) for each of m = 2, 3, 4, 5, 10 and 20. To better illustrate this result for large n and m, Fig. 2 graphs $-\log(1 - \mathrm{Eff}\,S_2)$ versus log(n). From either Fig. 1 or Fig. 2 the efficiency of S_1 can be obtained because it is equal to the efficiency of S_2 for n = 1, i.e., for log(n) = 0. Figures 1 and 2 illustrate that the efficiency of S_2 approaches 1 as n and/or m increase; hence, the variance approaches the lower bound for unbiased estimators.

MAXIMUM LIKELIHOOD ESTIMATOR S_M

The maximum likelihood estimator S_M given by:

$$S_M = \left(\frac{1}{n(m-1)}\sum_{i=1}^{n}\sum_{j=1}^{m}(z_j(i) - \bar{z}(i))^2\right)^{1/2} \tag{15}$$

is derived in Appendix D. The estimator S_M is exactly the same as S_2 except for a factor K.

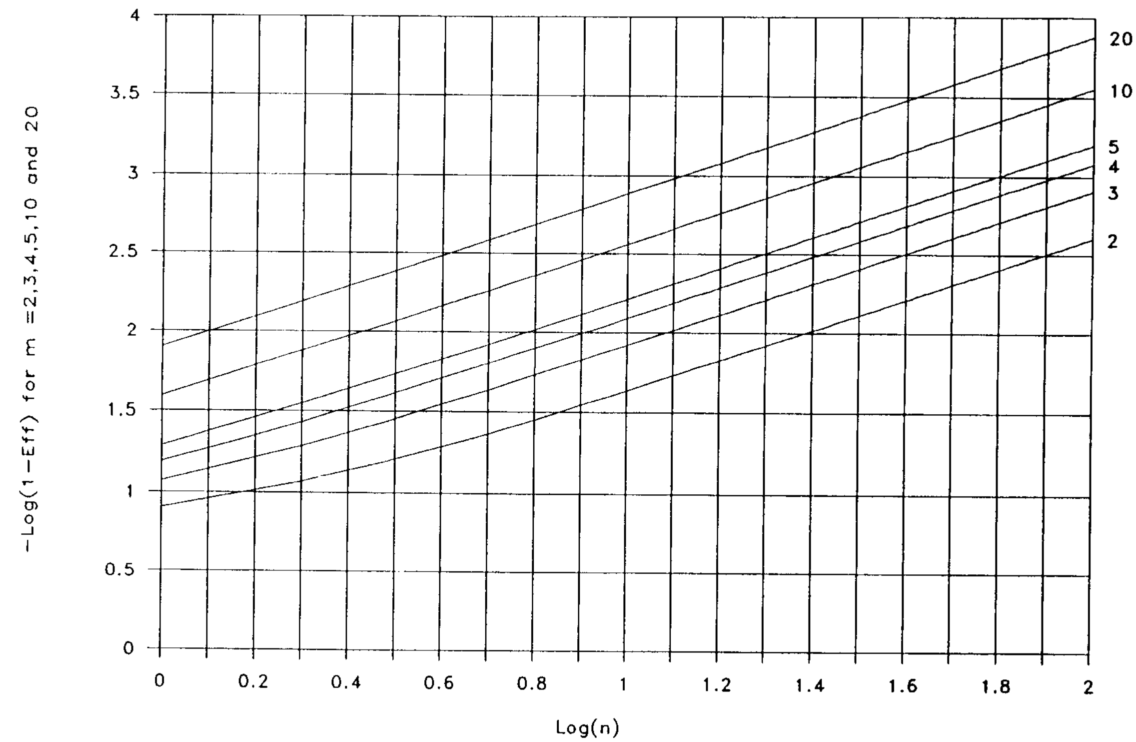

FIG. 2. −Log(1 − efficiency of S_2) versus log(n) for m = 2, 3, 4, 5, 10 and 20.

$$S_M = KS_2 \tag{16}$$

where

$$K = \frac{\Gamma((n(m-1)+1)/2)}{\Gamma(n(m-1)/2)}\left(\frac{2}{n(m-1)}\right)^{1/2} \tag{17}$$

Thus:

$$E[S_M] = KE[S_2] \tag{18}$$

$$\begin{aligned} \mathrm{Var}[S_M] &= K^2\mathrm{Var}[S_2] = \\ &= \sigma^2\left(1 - \frac{\Gamma^2((n(m-1)+1)/2)}{\Gamma^2(n(m-1)/2)}\,\frac{2}{n(m-1)}\right) \end{aligned} \tag{19}$$

From this, we see that the maximum likelihood estimator S_M is biased with a bias factor K which is always less than unity. The coefficient of variation of S_M is identical to that of S_2. The bias factor K approaches unity for large m and n as shown in Figs. 3 and 4 which illustrate K versus log(n) and −log(1 − K) versus log(n) for m = 2, 3, 4, 5, 10 and 20. These figures show that the bias of S_M becomes negligible as n and/or m increase. Even for m = 2, the bias factor is within 1% of unity for n ≥ 25.

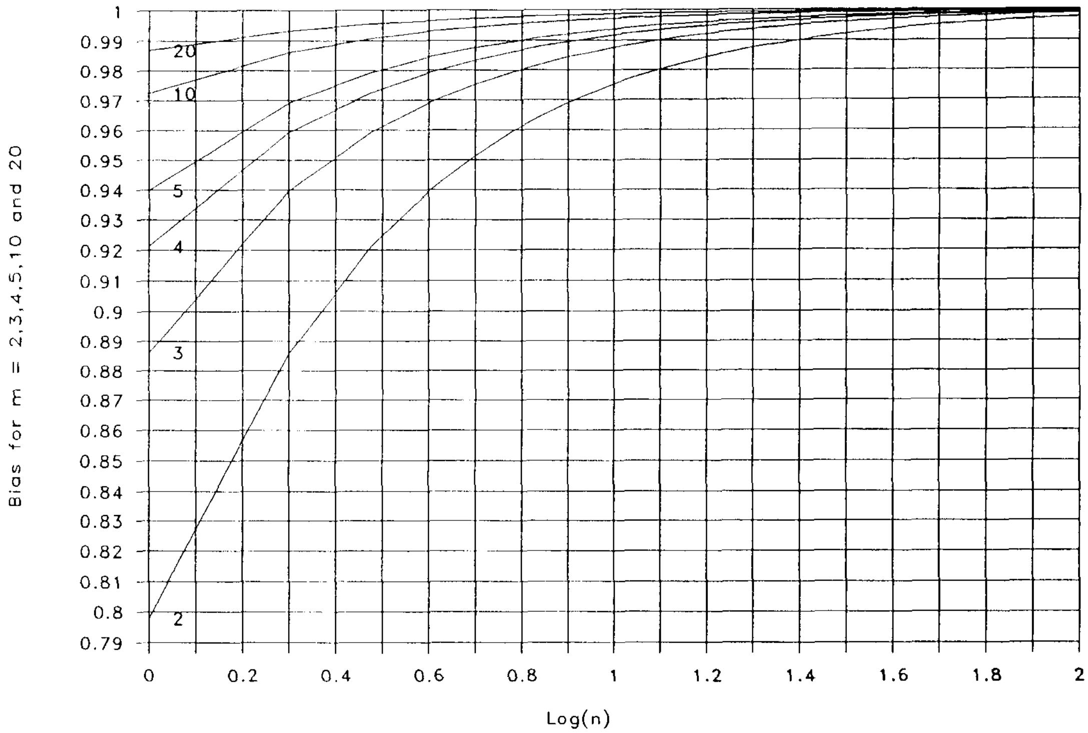

FIG. 3. Bias factor of estimator S_M versus log(n) for m = 2, 3, 4, 5, 10 and 20.

ESTIMATOR S_3

Estimator S_3 is defined in terms of the measurement order statistics.

$$S_3 = \frac{1}{n} \sum_{i=1}^{n} (z_{(q)}(i) - z_{(k)}(i)) \tag{20}$$

where k and q such that $1 \le k < q \le m$ are indices defining the particular order statistics used. The indices k and q define three blocks of the measurement domain, the block less than $z_{(k)}(i)$, the block between $z_{(k)}(i)$ and $z_{(q)}(i)$, and the block greater than $z_{(q)}(i)$.

The probability p_i contained in the block between the order statistics $z_{(k)}(i)$ and $z_{(q)}(i)$ has the beta probability density function given by:

$$f(p_i) = \frac{\Gamma(m + 1)}{\Gamma(q - k)\Gamma(m + 1 - q + k)} p_i^{q-k-1}(1 - p_i)^{m-q+k} \tag{21}$$

The expected value and variance of p_i are:

$$E[p_i] = \frac{q - k}{m + 1} \tag{22}$$

$$Var[p_i] = \frac{(q - k)(m + 1 - q + k)}{(m + 1)^2(m + 2)} \tag{23}$$

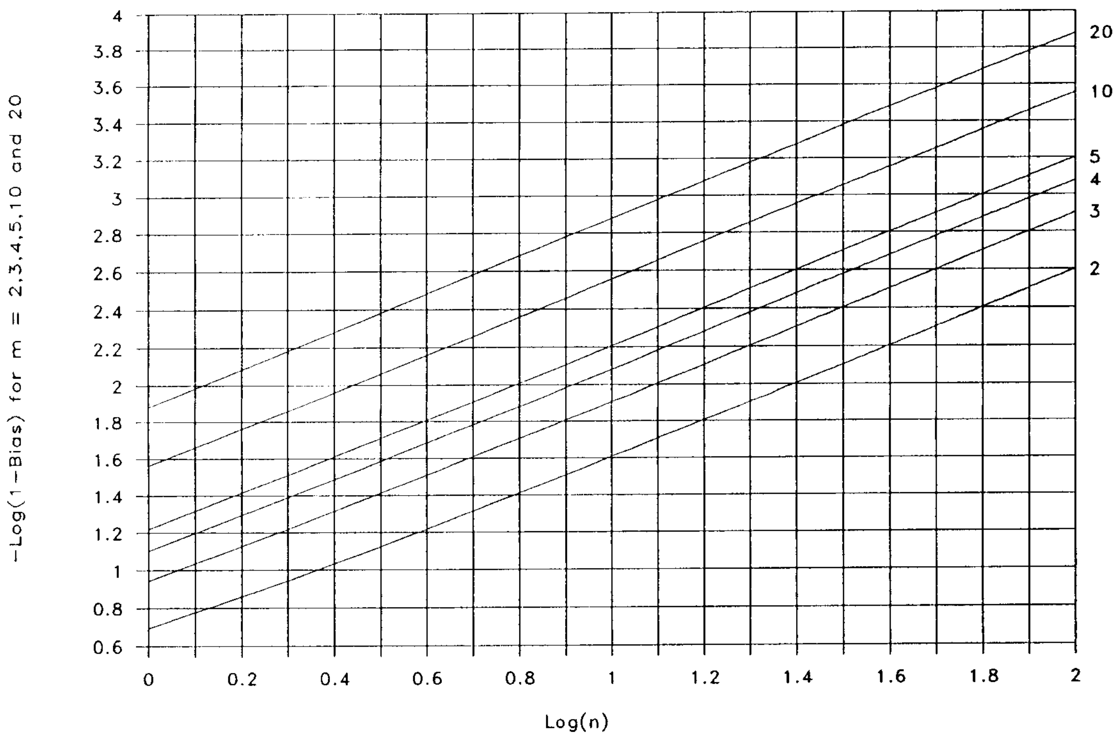

FIG. 4. $-$Log(1 $-$ bias factor of S_M) versus log(n) for m = 2, 3, 4, 5, 10 and 20.

By selecting k, q and m such that $q = 2k$ and $m = 3k - 1$, the expected value and variance of p_i become:

$$E[p_i] = \frac{1}{3} \tag{24}$$

$$Var[p_i] = \frac{2}{9(m + 2)} \tag{25}$$

The average $\bar{p}$ of p_i over the n items is:

$$\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i \tag{26}$$

with expected value and variance:

$$E[\bar{p}] = \frac{1}{3} \tag{27}$$

$$Var[\bar{p}] = \frac{2}{9n(m + 2)} \tag{28}$$

The coefficient of variation ratio for $\bar{p}$ is:

$$R_3 = \frac{(Var[\bar{p}])^{1/2}}{E[\bar{p}]} = \left(\frac{2}{n(m + 2)}\right)^{1/2} \tag{29}$$

The ratio R_3 applies to the average probability p contained in the n blocks whose boundaries are the order statistics used to define S_3. R_3 does not apply directly to the estimator S_3 itself. It should be noted that S_3 is not an estimator for the standard deviation σ.

AN IMPORTANT SPECIAL CASE, m = 2

For the special case, m = 2, we have:

$$S_1 = \frac{\sqrt{\pi}}{2n} \sum_{i=1}^{n} |z_1(i) - z_2(i)| \tag{30}$$

$$S_2 = \frac{\Gamma(n/2)}{2\Gamma((n+1)/2)} \left(\sum_{i=1}^{n} (z_1(i) - z_2(i))^2 \right)^{1/2} \tag{31}$$

$$S_M = \left(\frac{1}{2n} \sum_{i=1}^{n} (z_1(i) - z_2(i))^2 \right)^{1/2} \tag{32}$$

$$S_3 = \frac{1}{n} \sum_{i=1}^{n} |z_1(i) - z_2(i)| \tag{33}$$

This case is a practical one in that it involves measuring each of n items twice. Particularly in situations where measuring the item may affect it in some way that cumulatively could be significant, it is important to keep the measurement repetition number m as small as possible. The smallest value for obtaining a measure of residual error is m = 2.

We already determined in the general case that, except for a bias factor K, S_M and S_2 are the same; that is, $S_M = KS_2$. That relationship is also valid for the special case m = 2.

For this special case, we see also that except for a constant $2/\sqrt{\pi}$, S_3 and S_1 are equal.

$$S_3 = \frac{2}{\sqrt{\pi}} S_1, \qquad m = 2 \tag{34}$$

Therefore:

$$E[S_3] = \frac{2}{\sqrt{\pi}} E[S_1] = $$

$$= \frac{2}{\sqrt{\pi}} \sigma, \qquad m = 2 \tag{35}$$

AN EXAMPLE

At a sawmill in 1981, 25 pieces of 2 × 4 lumber were tested for modulus of elasticity (E) with a CLT Continuous Lumber Tester, a flatwise static tester, and a bending proof loader. The CLT is a high-speed, production-line machine that measures E on the flat with center loading over a 48-inch [1,219-mm] span. Span ends are defined by clamp rollers that approximate fixed end conditions. The flatwise static tester and the proof loader were home-built by the mill; the static tester measured E on the flat with third point loading on a 48-inch [1,219-mm] bending span, and the proof loader measured E on the edge with third point loading on approximately a 75-inch [1,905-mm] span. Two measurements of E were made on each piece of lumber by each machine. For these measurements, m = 2 and n = 25. Although North American CLTs use both the average (Average E) and the smallest (Low Point E) of the E

TABLE 3. *Example data and values for S_1, S_2, S_M and S_3.*

	Early data						Recent data	
	CLT AVG E		Static tester		Proof loader E		CLT AVG E	
Piece no.	Test 1	Test 2	Test 1	Test 2	Test 1	Test 2	Test 1	Test 2
1	1.65	1.60	1.56	1.56	1.71	1.67	2.27	2.27
2	1.65	1.60	1.51	1.53	1.61	1.61	1.79	1.78
3	1.98	1.96	1.88	1.90	2.15	2.13	1.61	1.60
4	1.35	1.32	1.15	1.19	1.39	1.36	1.46	1.46
5	1.91	1.90	1.78	1.83	2.09	2.00	1.78	1.78
6	1.49	1.47	1.35	1.39	1.55	1.56	2.01	2.00
7	1.37	1.36	1.22	1.24	1.40	1.38	1.70	1.70
8	1.65	1.65	1.51	1.48	1.88	1.85	1.47	1.47
9	1.98	1.98	1.85	1.86	2.02	2.07	1.58	1.57
10	1.88	1.87	1.73	1.76	2.17	2.17	1.81	1.80
11	1.90	1.90	1.85	1.85	1.90	1.91	1.79	1.78
12	1.95	1.97	1.83	1.86	1.90	1.91	1.68	1.67
13	2.11	2.11	1.97	1.97	2.13	2.26	1.69	1.70
14	1.40	1.40	1.24	1.25	1.34	1.33	2.00	2.02
15	1.46	1.45	1.35	1.34	1.42	1.43	2.07	2.06
16	2.00	1.98	1.90	1.88	2.00	2.02	1.93	1.93
17	1.86	1.86	1.72	1.70	2.02	2.00	1.73	1.74
18	1.48	1.49	1.49	1.45	1.61	1.61	1.66	1.66
19	2.21	2.21	2.12	2.06	2.17	2.17	1.76	1.76
20	1.82	1.82	1.85	1.72	1.93	1.91	1.56	1.56
21	1.63	1.63	1.50	1.45	1.63	1.67	1.47	1.47
22	1.59	1.58	1.46	1.41	1.64	1.71	1.29	1.31
23	1.92	1.92	1.86	1.81	1.91	1.95	1.29	1.30
24	1.85	1.85	1.85	1.78	1.75	1.76	1.44	1.44
25	1.74	1.74	1.65	1.62	1.96	1.96	1.39	1.40
26							1.83	1.82
27							1.69	1.69
28							1.88	1.87
29							1.87	1.87
30							1.97	1.97
S_1	0.0096		0.0294		0.0241		0.0053	
S_2	0.0129		0.0307		0.0291		0.0061	
S_M	0.0127		0.0304		0.0288		0.0061	
S_3	0.0108		0.0332		0.0272		0.0060	

measurements from all 48-inch spans along the length of each piece in determining E category, only the average is used in this example. The residual error estimators also can be used with Low Point E from the CLT (Logan 1991).

A number of recent tests on CLTs have been performed for the purpose of estimating residual error. Table 3 contains the data from the early tests and, for comparison, representative data from recent (1992) testing. These recent data consist of two E measurements (Average E) on each of 30 pieces of lumber from a well-tuned CLT running at 1,510 ft/min [460 m/min]. This is about 50% faster than for the earlier CLT E measurements. Table 3 also shows the S_1, S_2, S_M, and S_3 estimator values for each of the sets of data. By all four estimators, there is less residual error for the CLT than for the other two machines. The static tester and the proof loader have about the same residual error. The error estimates for the recent CLT tests are about half those from the earlier tests.

DISCUSSION AND CONCLUSIONS

Each of the estimators S_1, S_2, S_M, and S_3 has its advantages. Assuming a normal error distribution, S_1 and S_2 are both unbiased estimators of its standard deviation. S_2 is more efficient than S_1, but for the practical special case m = 2, S_1 is more easily computed. However, this advantage is much reduced with the use of modern computing technology.

The maximum likelihood estimator S_M is an easily computed, biased estimator of the standard deviation, and the bias becomes small as n increases (bias less than 1% for $m = 2$, $n \geq 25$).

The estimator S_3, which is the distance between specified measurement order statistics, is very easy to use. Except in the special case m = 2 where S_3 is related to S_1, the expected value and variance of S_3 are not stated. However, statistical information is given about the probability contained in blocks between order statistics.

Regarding the tradeoff between the number m of measurements made on each item and the number n of items measured, efficiency considerations weigh in favor of more m and less n. However, the difficulty of obtaining independent measurements and possible cumulative effects of repetitive measurements along with ease of computations are in favor of more n and less m. But, note that the efficiency of estimator S_1 is not affected by n, that is, it does not increase with n.

The recommendation here is to use the maximum likelihood estimator S_M for standard deviation because it is easily computed, and it is the "most likely" value for the standard deviation. Although S_M is biased, the bias becomes insignificant for reasonable sample sizes, and if desired, a known adjustment can be made to make the result unbiased (thereby converting S_M to S_2).

Example data accumulated more than 10 years ago and recently show improvements in S_M by about a factor of two for production-line E measuring equipment. The residual error reductions, even at higher operating speeds, are attributed to improvements in the equipment, equipment maintenance, and operating technique.

APPENDIX A: COMPUTATION OF E[S_1] AND VAR[S_1]

We assume that the measurement $z_j(i)$ can be broken into two parts as:

$$z_j(i) = x(i) + y_j(i) \tag{36}$$

where x(i) is the fixed but unknown property value for item i, and $y_j(i)$ is the residual error. We further assume that the values $y_j(i)$ are normally distributed and statistically independent with zero mean and variance σ^2. The estimator S_1 is:

$$S_1 = \frac{\Gamma((m-1)/2)}{\sqrt{2}\Gamma(m/2)} \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{m} (z_j(i) - \bar{z}(i))^2 \right)^{1/2} \tag{37}$$

where

$$\bar{z}(i) = \frac{1}{m} \sum_{j=1}^{m} z_j(i)) \tag{38}$$

The expression S_1 can be rewritten as:

$$S_1 = \frac{\Gamma((m-1)/2)}{\sqrt{2}n\Gamma(m/2)} \sum_{i=1}^{n} \sigma \left(\sum_{j=1}^{m} ((z_j(i) - \bar{z}(i))/\sigma)^2 \right)^{1/2} \tag{39}$$

where the quantity

$$r = \left(\sum_{j=1}^{m} ((z_j(i) - \bar{z}(i))/\sigma)^2 \right)^{1/2} \tag{40}$$

is distributed as a Rayleigh random variable (Miller 1964) with m − 1 degrees of freedom. The expected value and variance of r can be determined from its probability density function f(r);

$$\begin{aligned} f(r) &= 2r^{m-2} \frac{e^{-r^2/2}}{2^{(m-1)/2}\Gamma((m-1)/2)}, \qquad r \geq 0 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{41}$$

Then:

$$\begin{aligned} E[r] &= \int_0^\infty rf(r)\,dr = \\ &= \int_0^\infty \frac{2r^{m-1}e^{-r^2/2}}{2^{(m-1)/2}\Gamma((m-1)/2)}\,dr = \\ &= \int_0^\infty \frac{\sqrt{2}\Gamma(m/2)}{\Gamma((m-1)/2)} \frac{2r^{m-1}e^{-r^2/2}}{2^{m/2}\Gamma(m/2)}\,dr = \\ &= \frac{\sqrt{2}\Gamma(m/2)}{\Gamma((m-1)/2)} \end{aligned} \tag{42}$$

$$\begin{aligned} E[r^2] &= \int_0^\infty r^2 f(r)\,dr = \\ &= \int_0^\infty \frac{2r^m e^{-r^2/2}}{2^{(m-1)/2}\Gamma((m-1)/2)}\,dr = \\ &= \int_0^\infty \frac{2\Gamma((m+1)/2)}{\Gamma((m-1)/2)} \frac{2r^m e^{-r^2/2}}{2^{(m+1)/2}\Gamma((m+1)/2)}\,dr = \\ &= \frac{2\Gamma((m+1)/2)}{\Gamma((m-1)/2)} = \\ &= m - 1 \end{aligned} \tag{43}$$

$$\begin{aligned} \mathrm{Var}[r] &= E[r^2] - E^2[r] = \\ &= m - 1 - 2\frac{\Gamma^2(m/2)}{\Gamma^2((m-1)/2)} \end{aligned} \tag{44}$$

The expected value of S_1 is:

$$\begin{aligned} E[S_1] &= \frac{\Gamma((m-1)/2)}{\sqrt{2}n\Gamma(m/2)} \sum_{i=1}^{n} \sigma E[r] = \\ &= \sigma \end{aligned} \tag{45}$$

which proves that the estimator S_1 is unbiased. The variance of S_1 is:

$$\mathrm{Var}[S_1] = \frac{\Gamma^2((m-1)/2)}{2n^2\Gamma^2(m/2)} n\sigma^2\left(m - 1 - 2\frac{\Gamma^2(m/2)}{\Gamma^2((m-1)/2)}\right) =$$

$$= \frac{\sigma^2}{n}\left(((m-1)/2)\frac{\Gamma^2((m-1)/2)}{\Gamma^2(m/2)} - 1\right) \tag{46}$$

APPENDIX B: COMPUTATION OF E[S_2] AND VAR[S_2]

Start with the same assumptions as in Appendix A and rewrite S_2 as:

$$S_2 = \frac{\Gamma(n(m-1)/2)\sigma}{\sqrt{2}\Gamma((n(m-1)+1)/2)}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}((z_j(i) - \bar{z}(i))/\sigma)^2\right)^{1/2} \tag{47}$$

Each quantity

$$r_i = \left(\sum_{j=1}^{m}((z_j(i) - \bar{z}(i))/\sigma)^2\right)^{1/2}, \qquad i = 1, 2, \ldots, n \tag{48}$$

is a Rayleigh random variable with $m - 1$ degrees of freedom. Therefore, r defined by:

$$r = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} r_i^2\right)^{1/2} \tag{49}$$

is a Rayleigh variate with $n(m - 1)$ degrees of freedom. From Appendix A it follows that:

$$E[r] = \sqrt{2}\frac{\Gamma((n(m-1)+1)/2)}{\Gamma(n(m-1)/2)} \tag{50}$$

$$\mathrm{Var}[r] = n(m-1) - 2\frac{\Gamma^2((n(m-1)+1)/2)}{\Gamma^2(n(m-1)/2)} \tag{51}$$

Thus:

$$E[S_2] = \sigma \tag{52}$$

$$\mathrm{Var}[S_2] = \frac{\Gamma^2(n(m-1)/2)\sigma^2}{2\Gamma^2((n(m-1)+1)/2)}\left(n(m-1) - 2\frac{\Gamma^2((n(m-1)+1)/2)}{\Gamma^2(n(m-1)/2)}\right) =$$

$$= \sigma^2\left(\frac{n(m-1)}{2}\frac{\Gamma^2(n(m-1)/2)}{\Gamma^2((n(m-1)+1)/2)} - 1\right) \tag{53}$$

APPENDIX C: LOWER BOUND FOR UNBIASED ESTIMATORS

The theoretical lower bound for the variance of an unbiased estimator S for the parameter σ is known (Hogg and Craig 1965) to be:

$$\mathrm{MinVar} = \frac{1}{nE\left[\left(\frac{\partial \ln f(x;\sigma)}{\partial\sigma}\right)^2\right]} \tag{54}$$

where $x_1, x_2, \ldots, x_n$ is a random sample of size n from the distribution having density functions $f(x; \sigma)$ which is a function of the parameter σ. The domain of x cannot depend on σ.

For the present case, we define x_i as:

$$x_i = \sigma r_i, \qquad i = 1, 2, \ldots, n \tag{55}$$

where each r_i is a Rayleigh variate having $m - 1$ degrees of freedom as in Appendix B. With the change of variable $x = \sigma r$, the density function (see Appendix A) is:

$$f(x; \sigma) = \frac{x^{m-2}\exp(-(x/\sigma)^2/2)}{2^{(m-3)/2}\Gamma((m-1)/2)}\left(\frac{1}{\sigma}\right)^{m-1} \tag{56}$$

Carrying out the indicated operations, we obtain:

$$E\left[\left(\frac{\partial \ln f(x; \sigma)}{\partial \sigma}\right)^2\right] = \frac{1}{\sigma^2}\left(\frac{E[x^4]}{\sigma^4} - 2(m-1)\frac{E[x^2]}{\sigma^2} + (m-1)^2\right) \tag{57}$$

Recognizing that $E[x^2] = \sigma^2 E[r^2]$ and $E[x^4] = \sigma^4 E[r^4]$, using $E[r^2] = m - 1$ from Appendix A and performing similar operations to obtain $E[r^4] = (m + 1)(m - 1)$, the above reduces to:

$$E\left[\left(\frac{\partial \ln f(x; \sigma)}{\partial \sigma}\right)^2\right] = \frac{2(m-1)}{\sigma^2} \tag{58}$$

and finally:

$$\mathrm{MinVar} = \frac{\sigma^2}{2n(m-1)} \tag{59}$$

APPENDIX D: MAXIMUM LIKELIHOOD ESTIMATOR S_M

From Appendix B we have that:

$$r_i = \left(\sum_{j=1}^{m} ((z_j(i) - \bar{z}(i))/\sigma)^2\right)^{1/2} \tag{60}$$

is a Rayleigh variate with $m - 1$ degrees of freedom. Hence, its density function is:

$$f(r_i) = 2r_i^{m-1}\frac{\exp(-r_i^2/2)}{2^{(m-1)/2}\Gamma((m-1)/2)} \tag{61}$$

and the density function of $u_i = \sigma r_i$ is:

$$f(u_i) = \frac{u_i^{m-2}\exp(-(u_i/\sigma)^2/2)}{2^{(m-3)/2}\Gamma((m-1)/2)}\left(\frac{1}{\sigma}\right)^{m-1} \tag{62}$$

The joint density function corresponding to an independent sample of n items, each measured m times is:

$$f(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n} f(u_i) \tag{63}$$

The maximum likelihood estimator S_M is the value of σ which maximizes $f(u_1, u_2, \ldots, u_n)$ or equivalently its logarithm $\ln[f(u_1, u_2, \ldots, u_n)]$. By setting to zero the derivative of $\ln[f(u_1, u_2, \ldots, u_n)]$ with respect to σ, we obtain:

$$S_M = \left(\frac{1}{n(m-1)}\sum_{i=1}^{n}\sum_{j=1}^{m}(z_j(i) - \bar{z}(i))^2\right)^{1/2} \tag{64}$$

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