PLANE WAVE PROPAGATION AND INTERNAL INSTABILITY IN FINELY LAYERED MEDIA

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ABSTRACT

An exact elasticity solution for transverse waves is presented for an infinite periodic layered medium that is subjected to initial stress and that is incrementally elastic. Exact results are compared with various approximate theories. It appears that the microstructure approximate method may be sufficiently accurate to permit use in problems of acoustic propagation and impact loading. Internal instability of the layered medium is treated as a special case of the general dynamic problem.

INTRODUCTION

The development of analytical methods for predicting the mechanical behavior of composite materials has received considerable attention recently. The physical model of a finely layered periodic medium has been frequently employed in the description of composite material behavior related to dynamic response.

Wood material can be modeled as finely layered medium at various structural levels: at the level of the cell wall, at the fiber-to-fiber level, and at the level of the growth increments. It is felt that these characteristic structural features have a considerable influence on such mechanical phenomena as acoustical attenuation and the response to impact loading.

A general philosophy of approach to problems of predicting the mechanical response of heterogeneous media such as solid wood has been presented by Perkins (1972); however, in this work numerical results were not available. The present work presents an exact elasticity solution for the problem of continuous plane wave propagation in an incrementally deformed layered medium. The results of the exact solution are presented and compared extensively with the results of the various approximate theories presented in Perkins (1972). While the information presented is restricted to the case of continuous plane wave propagation, the results demonstrate the possible utility and the limitations of the approximate procedures. These results pave the way for the development of analyses of more complex wave phenomena where the approximate theories can be utilized to describe the modal form and frequency relationships pertinent to wave propagation. It is expected that this methodology can be profitably employed to describe certain forms of acoustic or impact loading response in finely layered media.

Another feature of the present work is the consideration of internal instability of a laminated medium. From a practical viewpoint, it is believed that in some situations the compressive strength of a layered material can be described as a manifestation of internal instability. Biot (1963, 1967) analyzed this problem extensively, although his results were restricted in that he always assumed the materials to be incompressible. We have presented the exact solutions and the microstructure and couple stress approaches to this problem and have compared such solutions. Also presented in this comparison are numerical results based on a recent work by Kiusalaas and Jaunzemis (in press). In this work, the reinforcement is treated as a plate and the

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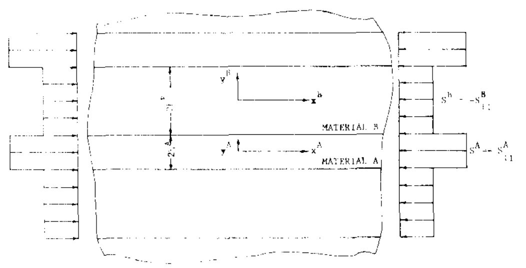

FIG. 1. Layered medium subjected to a state of initial homogeneous strain.

matrix is treated according to the micro-structure approach.

ANALYTICAL SOLUTIONS

A layered medium of infinite extent is assumed to be in a state of initial homogeneous strain with the principal directions of initial strain in the principal planes of the layering (Fig. 1). The layered structure is assumed to be composed of alternating layers of two different incrementally orthotropic media. Attention is restricted to the case of plane strain disturbances. We consider here two problems: (a) the propagation of a transverse plane wave where the direction of wave propagation is the same as the direction of initial stress, and (b) elastic stability of the layered medium.

The initial stresses in the two materials are different because their elastic moduli are different. The initial strain, however, is homogeneous.

At the middle of each layer a coordinate system is set up as shown in Fig. 1.

Fundamental relations

Following Biot's (1965) incremental deformation theory, the initial stresses are denoted by S_{11}, S_{22}, S_{33}. The displacement components measured from the state of initial stress are denoted by u_1, u_2, u_3. The infinitesimal (first-order) incremental strain components are

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (1)$$

where $j_{,i} \equiv \partial/\partial x_i$. The first order local rotation in the (x_1, x_2) plane is given by

$$\phi = \frac{1}{2}(u_{2,1} - u_{1,2}) \qquad (2)$$

Incremental stresses referred to incrementally deformed areas and axes that rotate locally with the medium are denoted by s_{ij}. Equations of motion in terms of these stresses for plane deformation with $S_{22} = S_{33} = 0$ are:

$$\begin{aligned} \frac{\partial s_{11}}{\partial x_1} + \frac{\partial s_{12}}{\partial x_2} + S_{11}\frac{\partial \phi}{\partial x_2} &= \rho \ddot{u}_1 \\ \frac{\partial s_{12}}{\partial x_1} + \frac{\partial s_{22}}{\partial x_2} + S_{11}\frac{\partial \phi}{\partial x_1} &= \rho \ddot{u}_2 \end{aligned} \qquad (3)$$

where a dot denotes differentiation with respect to time.

For an incrementally orthotropic elastic medium the incremental stresses and strains are assumed to be linearly related by

$$\begin{aligned} s_{11} &= B_{11}e_{11} + B_{12}e_{22} \\ s_{22} &= B_{21}e_{11} + B_{22}e_{22} \\ s_{12} &= 2Qe_{12} \end{aligned} \qquad (4)$$

Using (1), (2), and (4) in (3), we obtain the displacement equations of motion:

$$\begin{aligned} a_{11}u_{1,11} + b_{12}u_{2,12} + a_{22}u_{1,22} &= \rho\ddot{u}_1 \\ b_{22}u_{2,22} + a_{12}u_{1,12} + b_{11}u_{2,11} &= \rho\ddot{u}_2 \end{aligned} \qquad (5)$$

where

$$\begin{aligned} &a_{11} = B_{11}; \quad b_{12} = B_{12} + Q - \frac{1}{2}S \\ &a_{22} = Q + \frac{1}{2}S; \quad b_{22} = B_{22} \\ &a_{12} = B_{21} + Q + \frac{1}{2}S \\ &b_{11} = Q - \frac{1}{2}S; \quad S = -S_{11} \end{aligned} \qquad (6)$$

Boundary stresses in the directions of x_1 and x_2, respectively, are

$$\begin{aligned} f_{x_1} &= (S_{11} + s_{11} + S_{11}e_{22})\,\mathrm{Cos}(\underline{n}, x_1) \\ &\quad + (s_{12} - S_{11}e_{21})\,\mathrm{Cos}(\underline{n}, x_2) \\ f_{x_2} &= (s_{12} + S_{11}\phi)\,\mathrm{Cos}(\underline{n}, x_1) \\ &\quad + (s_{22})\,\mathrm{Cos}(\underline{n}, x_2) \end{aligned} \qquad (7)$$

where $(\underset{\sim}{n}, x_i)$ is the angle between the outer normal to the boundary $\underset{\sim}{n}$ and the coordinate axis x_i. Using (1), (2), (4), and (6) in (7), we have the boundary stresses in terms of displacements:

$$
\begin{aligned}
f_{x_1} &= [-S + a_{11}u_{1,1} + (b_{12} - a_{22})u_{2,2}] \\
&\quad \cdot \cos(\underset{\sim}{n}, x_1) + a_{22}(u_{1,2} + u_{2,1})\cos(\underset{\sim}{n}, x_2) \\
f_{x_2} &= [(a_{22} + \tfrac{1}{2}S)u_{1,2} + (b_{11} - \tfrac{1}{2}S)u_{2,1}] \\
&\quad \cdot \cos(\underset{\sim}{n}, x_1) + [\, b_{22}u_{2,2} + (a_{12} \\
&\quad - a_{22})u_{1,1}] \cos(\underset{\sim}{n}, x_2)
\end{aligned} \tag{8}
$$

Elasticity solution

(a) *Wave propagation*: Denoting the coordinates x_i and the displacements u_i for $i = 1,2$ by x,y and u,v, respectively, we seek a solution to (5) in the form

$$
\begin{aligned}
u &= U(y) \exp i(ax - \omega t) \\
v &= V(y) \exp i(ax - \omega t)
\end{aligned} \tag{9}
$$

where $2\pi/a$ represents the wavelength and $\omega/a = c$ represents the phase velocity of the propagating wave and now $i = \sqrt{-1}$.

Substitution of (9) in (5) leads to a pair of simultaneous linear ordinary differential equations in U and V:

$$
\begin{aligned}
a_{22}U'' + (\rho\omega^2 - a^2 a_{11})U + b_{12}iaV' &= 0 \\
b_{22}V'' + (\rho\omega^2 - a^2 b_{11})V + a_{12}iaU' &= 0
\end{aligned} \tag{10}
$$

where a prime denotes differentiation with respect to y. The general solution to (10) can be written in the form:

$$
\begin{aligned}
U &= (\rho\omega^2 - b_{22}\beta^2 - a^2 b_{11})[F_1 \cos\beta y + iF_2 \sin\beta y] \\
&\quad + b_{12}a\delta[F_3 \cos\delta y + iF_4 \sin\delta y] \\
V &= a_{12}a\beta[F_2 \cos\beta y + iF_1 \sin\beta y] \\
&\quad + (\rho\omega^2 - a_{22}\delta^2 - a^2 a_{11})[F_4 \cos\delta y + iF_3 \sin\delta y]
\end{aligned} \tag{11}
$$

where the F_i are arbitrary constants and β^2 and δ^2 are the two values of λ^2 in the characteristic equation:

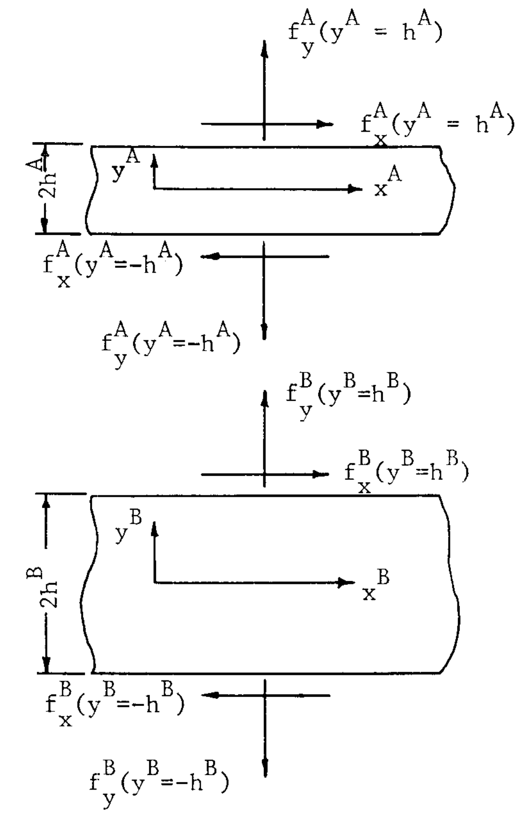

FIG. 2. Boundary stresses at the interfaces between layers.

$$
\begin{aligned}
&a_{22}b_{22}\lambda^4 + [\rho\omega^2(a_{22} + b_{22}) + a^2(a_{12}b_{12} \\
&- a_{11}b_{22} - a_{22}b_{11})]\lambda^2 + (\rho\omega^2 - a^2 a_{11})(\rho\omega^2 \\
&- a^2 b_{11}) = 0
\end{aligned} \tag{12}
$$

Equations of the form (11) apply for each of the layers in the infinite layered structure. Since the layered structure is assumed to have a repeating pattern consisting of only two different layers, it is sufficient to consider one representative set of two layers as in Fig. 2. We will use superscripts A and B to distinguish between quantities pertaining to the two materials.

We have eight constants F_j^A, F_j^B $(j = 1, 2, 3, 4)$ to be determined by boundary conditions for each layer. These are conditions of continuity of displacements and tractions at the interfaces:

$$
\begin{aligned}
u^A(y^A=-h^A) &= u^B(y^B=h^B) \\
v^A(y^A=-h^A) &= v^B(y^B=h^B) \\
u^A(y^A=h^A) &= u^B(y^B=-h^B) \\
v^A(y^A=h^A) &= v^B(y^B=-h^B)
\end{aligned}
\qquad (13)
$$

$$
\begin{aligned}
f_x^A(y^A=-h^A) &= -f_x^B(y^B=h^B) \\
f_y^A(y^A=-h^A) &= -f_y^B(y^B=h^B) \\
f_x^A(y^A=h^A) &= -f_x^B(y^B=-h^B) \\
f_y^A(y^A=h^A) &= -f_y^B(y^B=-h^B)
\end{aligned}
\qquad (14)
$$

Attention should be called to the signs in (14). Figure 2 is self-explanatory.

The conditions of continuity (13) and (14) provide a system of eight linear homogeneous algebraic equations for the determination of the eight constants F_j^A, F_j^B ($j=1,2,3,4$). It is possible to manipulate these equations so that we have four equations in $F_{1,3}^{A,B}$ alone and four equations in $F_{2,4}^{A,B}$ alone. If we set $F_{1,3}^{A,B}\equiv 0$, the displacement in the direction of wave propagation when averaged over the layer thickness becomes zero for each layer. Thus the propagating wave is an essentially transverse wave. This definition of a transverse wave in a layered medium follows the procedure of Rytov (1956) as reported in Brekhovskikh (1960). In this case we are left with four equations in $F_{2,4}^{A,B}$:

$$
[g][F] \equiv \begin{bmatrix} g_{11}^A & g_{12}^A & g_{11}^B & g_{12}^B \\ g_{21}^A & g_{22}^A & g_{21}^B & g_{22}^B \\ g_{31}^A & g_{32}^A & g_{31}^B & g_{32}^B \\ g_{41}^A & g_{42}^A & g_{41}^B & g_{42}^B \end{bmatrix} \begin{bmatrix} F_2^A \\ F_4^A \\ F_2^B \\ F_4^B \end{bmatrix} = [0] \qquad (15)
$$

where [0] is the null column matrix, and the elements of $[g]$ are[3]

$$
\begin{aligned}
g_{11}^{A,B} &= \left[\rho^{A,B}\omega^2 - b_{22}^{A,B}(\beta^{A,B})^2 - b_{11}^{A,B}a^2\right]\mathrm{Sin}(\beta^{A,B}h^{A,B}) \\
g_{12}^{A,B} &= \left[b_{12}^{A,B}a\delta^{A,B}\right]\mathrm{Sin}(\delta^{A,B}h^{A,B}) \\
g_{21}^{A,B} &= \pm\left[a_{12}^{A,B}a\beta^{A,B}\right]\mathrm{Cos}(\beta^{A,B}h^{A,B}) \\
g_{22}^{A,B} &= \pm\left[\rho^{A,B}\omega^2 - a_{22}^{A,B}(\delta^{A,B}) - a_{11}^{A,B}a^2\right]\mathrm{Cos}(\delta^{A,B}h^{A,B}) \\
g_{31}^{A,B} &= \pm a_{22}^{A,B}\beta^{A,B}\Big[\rho^{A,B}\omega^2 - b_{22}^{A,B}(\beta^{A,B})^2 \\
&\qquad + (a_{12}^{A,B} - b_{11}^{A,B})a^2\Big]\mathrm{Cos}(\beta^{A,B}h^{A,B}) \qquad (16) \\
g_{32}^{A,B} &= \pm a_{22}^{A,B}a\Big[\rho^{A,B}\omega^2 + (b_{12}^{A,B} - a_{22}^{A,B})(\delta^{A,B})^2 \\
&\qquad - a_{11}^{A,B}a^2\Big]\mathrm{Cos}(\delta^{A,B}h^{A,B}) \\
g_{41}^{A,B} &= a\Big[(a_{12}^{A,B} - a_{22}^{A,B})(\rho^{A,B}\omega^2 - b_{22}^{A,B}(\beta^{A,B})^2 \\
&\qquad - b_{11}^{A,B}a^2 + b_{22}^{A,B}a_{12}^{A,B}(\beta^{A,B})^2\Big]\mathrm{Sin}(\beta^{A,B}h^{A,B}) \\
g_{42}^{A,B} &= \delta^{A,B}\Big[b_{22}^{A,B}(\rho^{A,B}\omega^2 - a_{22}^{A,B}(\delta^{A,B})^2 - a_{11}^{A,B}a^2) \\
&\qquad + (a_{12}^{A,B} - a_{22}^{A,B})b_{12}^{A,B}a^2\Big]\mathrm{Sin}(\delta^{A,B}h^{A,B})
\end{aligned}
$$

A nontrivial solution obtains for combinations of circular frequency ω and wave number a when

$$
\det[g] = 0 \qquad (17)
$$

Equation (17), the characteristic frequency equation, represents the dependence of wave number, or if one wishes, the phase velocity, of the propagating wave on the wave frequency. Thus (17) represents the dispersion relation for transverse plane waves propagating through the layered medium in the direction of the plane of the layers aligned with the direction of the initial stress.

(b) *Elastic stability*: Normally the solution for sinusoidal buckling modes could be obtained from the wave propagation solution just described by dropping the time-dependent terms, i.e., by setting $\omega=0$. For the present case, however, a peculiar situation occurs.

We seek solutions to (5) in the form:

[3] Use upper sign for material A and lower sign for material B.

$$u = U(y)\operatorname{Cos} ax$$
$$v = V(y)\operatorname{Sin} ax \tag{18}$$

Substituting (18) into (5), we arrive at two ordinary simultaneous linear differential equations in U and V:

$$a_{22}U'' - a^2 a_{11}U + b_{12}aV' = 0$$
$$b_{22}V'' - a^2 b_{11}V - a_{12}aU' = 0 \tag{19}$$

It turns out that the characteristic equation

$$a_{22}b_{22}\lambda^4 + (a_{12}b_{12} - b_{22}a_{11} - a_{22}b_{11})a^2\lambda^2 + a_{11}b_{11}a^4 = 0 \tag{20}$$

has repeated roots $\lambda = \pm a, \pm a$. (This can be seen by using (6) and the expressions for B_{ij} and Q provided in Section 3 in (20) and solving for λ.)

For this case the general solution to (19) can be written in the form:

$$U = b_{12}\left[F_1 \operatorname{Cosh} ay + F_2 \operatorname{Sinh} ay + F_3\, ay \operatorname{Sinh} ay + F_4\, ay \operatorname{Cosh} ay\right]$$
$$V = \left[(a_{11} - a_{22})F_2 - (a_{11} + a_{22})F_4\right]\operatorname{Cosh} ay + \left[(a_{11} - a_{22})F_1 - (a_{11} + a_{22})F_3\right]\operatorname{Sinh} ay + (a_{11} - a_{22})F_4\, ay \operatorname{Sinh} ay + (a_{11} - a_{22})F_3\, ay \operatorname{Cosh} ay \tag{21}$$

From here on the procedure is exactly the same as for the wave problem. Again we have eight constants F_j^A, F_j^B ($j = 1, 2, 3, 4$) to be determined by the continuity conditions (13) and (14), which lead to eight linear homogeneous algebraic equations in the eight constants. As before, these equations can be manipulated to yield four equations in $F^{A,B}_{1,3}$ and four equations in $F^{A,B}_{2,4}$. Setting $F^{A,B}_{1,3} \equiv 0$ leaves the four constants $F^{A,B}_{2,4}$, which describe a buckling mode referred to in the literature as the "antisymmetric" buckling mode. The four equations in $F^{A,B}_{2,4}$ are:

$$[h][F] \equiv \begin{bmatrix} h_{11}^A & h_{12}^A & h_{11}^B & h_{12}^B \\ h_{21}^A & h_{22}^A & h_{21}^B & h_{22}^B \\ h_{31}^A & h_{32}^A & h_{31}^B & h_{32}^B \\ h_{41}^A & h_{42}^A & h_{41}^B & h_{42}^B \end{bmatrix} \begin{bmatrix} F_2^A \\ F_4^A \\ F_2^B \\ F_4^B \end{bmatrix} = [0] \tag{22}$$

where the elements of $[h]$ are[4]

$$h_{11}^{A,B} = b_{12}^{A,B} \operatorname{Sinh}(ah^{A,B})$$
$$h_{12}^{A,B} = b_{12}^{A,B}\, ah^{A,B} \operatorname{Cosh}(ah^{A,B})$$
$$h_{21}^{A,B} = \pm(a_{11}^{A,B} - a_{22}^{A,B})\operatorname{Cosh}(ah^{A,B})$$
$$h_{22}^{A,B} = \pm(a_{11}^{A,B} - a_{22}^{A,B})\, ah^{A,B} \operatorname{Sinh}(ah^{A,B}) \mp (a_{11}^{A,B} + a_{22}^{A,B})\operatorname{Cosh}(ah^{A,B})$$
$$h_{31}^{A,B} = \pm a_{22}^{A,B}(b_{12}^{A,B} + a_{11}^{A,B} - a_{22}^{A,B})\operatorname{Cosh}(ah^{A,B})$$
$$h_{32}^{A,B} = \pm a_{22}^{A,B}(b_{12}^{A,B} + a_{11}^{A,B} - a_{22}^{A,B})\, ah^{A,B} \cdot \operatorname{Sinh}(ah^{A,B}) \pm a_{22}^{A,B}(b_{12}^{A,B} - a_{11}^{A,B} - a_{22}^{A,B})\operatorname{Cosh}(ah^{A,B})$$
$$h_{41}^{A,B} = \left[b_{22}^{A,B}(a_{11}^{A,B} - a_{22}^{A,B}) - b_{12}^{A,B}(a_{12}^{A,B} - a_{22}^{A,B})\right] \cdot \operatorname{Sinh}(ah^{A,B})$$
$$h_{42}^{A,B} = \left[b_{22}^{A,B}(a_{11}^{A,B} - a_{22}^{A,B}) - b_{12}^{A,B}(a_{12}^{A,B} - a_{22}^{A,B})\right] \cdot ah^{A,B}\operatorname{Cosh}(ah^{A,B}) - 2b_{22}^{A,B}a_{22}^{A,B}\operatorname{Sinh}(ah^{A,B}) \tag{23}$$

A nontrivial solution in $F^{A,B}_{2,4}$ obtains when

$$\det[h] = 0 \tag{24}$$

Equation (24) represents the dependence of wave number "a" or the wavelength $L = 2\pi/a$, on the initial stress "S," and we shall call it the stability equation.

Microstructure solution

(a) *Wave propagation*: The microstructure theory including initial stress was applied to the propagation of transverse waves in a periodic laminated medium by Perkins

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[4] See Footnote 3.

(1972). The following dispersion relation (in the present notation) was arrived at:

$$\left[(\rho^A h^A+\rho^B h^B)\, c^2-(b_{11}^A h^A+b_{11}^B h^B)\right] \cdot\left[\frac{a^2}{3}\left\{(\rho^A h^A+\rho^B h^B)\, c^2-(a_{11}^A h^A+a_{11}^B h^B)\right\} -\frac{a_{22}^A}{h^A}+\frac{a_{22}^B}{h^B}\right] -(a_{22}^A-a_{22}^B)^2=0 \quad (25)$$

In deriving this relation, the effect of initial stress in both materials of the composite was taken into account. Also, linear microdisplacement fields in both materials were considered.

(b) *Elastic stability*: Setting $c=0$ in (25) yields the stability equation:

$$(b_{11}^A h^A+b_{11}^B h^B)\left[\frac{a^2}{3}(a_{11}^A h^A+a_{11}^B h^B)+\left(\frac{a_{22}^A}{h^A}+\frac{a_{22}^B}{h^B}\right)\right] -(a_{22}^A-a_{22}^B)^2=0 \quad (26)$$

Couple stress solution

(a) *Wave propagation*: Perkins (1972) extended Biot's (1965) incremental deformation theory to include couple stresses for the case of plane strain and sought solutions to the equations of motion in the form

$$u=0 \quad ; \quad v=V_0 \exp\left[ia(x-ct)\right] \quad (27)$$

where V_0 is a constant and represents the amplitude of the transverse wave. This led to a dispersion relation, which in the present notation is

$$\rho c^2=b_{11}+a^2K_1 \quad (28)$$

where K_1 is the couple stress coefficient relating the couple stress m_x acting on a face initially parallel to the y-axis to the curvature $(\partial\phi/\partial x)$ by $m_x=4K_1(\partial\phi/\partial x)$.

Equation (28) represents the dispersion relation for a homogeneous medium. Note that if couple stresses were not taken into account, there would be no dispersion for the displacement field given by (27). (28) can be applied to a laminated medium if the quantities in the equation represent "effective values" (Biot 1963), i.e., if

$$\rho=\alpha^A\rho^A+\alpha^B\rho^B \; ; \; b_{11}=\alpha^A b_{11}^A+\alpha^B b_{11}^B \quad (29)$$

where

$$\alpha^{A,B}=\frac{h^{A,B}}{h^A+h^B}$$

Biot (1967) proposed an expression for the coefficient K_1 of the laminated medium assuming that the materials are incompressible. The same procedure is readily adaptable to compressible materials,[5] and the following expression obtains:

$$K_1^1=\frac{1}{6}\,\frac{\left[(h^A)^2a_{11}^A-(h^B)^2a_{11}^B\right]}{\left(\frac{a_{22}^A}{h^A}+\frac{a_{22}^B}{h^B}\right)}\cdot\frac{(a_{22}^A-a_{22}^B)}{(h^A+h^B)} \quad (30)$$

By reducing the field equations of the microstructure theory to those of the couple stress theory and comparing these with the field equations of the couple stress theory, Perkins (1972) deduced the following expression for K_1 of the laminated medium:

$$K_1^2=\frac{1}{3}\,\frac{(a_{11}^A h^A+a_{11}^B h^B)}{\left(\frac{a_{22}^A}{h^A}+\frac{a_{22}^B}{h^B}\right)^2}\cdot\frac{(a_{22}^A-a_{22}^B)^2}{(h^A+h^B)} \quad (31)$$

Both the above expressions have been used in numerical computations.

(b) *Elastic Stability*: Setting $c=0$ in (28) gives the stability equation

$$b_{11}+a^2K_1=0 \quad (32)$$

Note again that if couple stresses were not considered, the stability equation would be independent of the wavelength, for the displacement field given by (27).[6]

Again, (32) is applicable to a laminated medium if the coefficients stand for those of the laminated medium.

[5] See, for example, Perkins (1972).

[6] Biot (1963) obtained a stability equation dependent upon wavelength without considering couple stresses. However, the displacement field was different from (27). He also extended his own analysis to include couple stresses in (1967).

Another solution for elastic stability

Kiusalaas and Jaunzemis (in press) presented a continuum theory for internal buckling of a laminated medium satisfying the inequalities

$$E^A h^A >> E^B h^B \text{ and } h^A << h^B \qquad (33)$$

(We will occasionally refer to materials A and B as the reinforcement and matrix, respectively.) They treated the reinforcement as plates and considered linear microdisplacement fields in the matrix. Furthermore, they neglected the energy due to initial strain in the matrix as being small compared with the prestrain energy in the reinforcement, an assumption justifiable by the first inequality above. Their stability equation for the "antisymmetric" or "shear" buckling mode in our notation is

$$p = \frac{1-(\nu^A)^2}{1+\nu^B} \, \frac{E^B(h^A+h^B)}{E^A\,(2h^A)} + \frac{1}{3}\left(\frac{h^A}{h^A+h^B}\right)^2 \cdot (h^A+h^B)^2 a^2 \qquad (34)$$

where p is the strain before buckling and is assumed to be small, i.e., $p << 1$.

THE INCREMENTAL MODULI B_{ij} AND Q

In what follows, repeated indices are not to be summed over unless a summation sign so implies.

Let the natural stress-free state of a material (A or B) be characterized by coordinates X_i. Now let the material undergo a homogeneous deformation so that a generic point has coordinates given by

$$x_i = \Gamma_i X_i \qquad (35)$$

An increment in the Γ_i's gives rise to incremental strains

$$e_{ii} = \frac{d\Gamma_i}{\Gamma_i} \qquad (36)$$

It is in general possible to express the stresses S_{ii} in terms of the "stretches" Γ_i. We assume the general form of these relations as follows:

$$S_{ii} = S_{ii}(\Gamma_1, \Gamma_2, \Gamma_3) \qquad (37)$$

The incremental stresses s_{ii} may be identified with the differentials of these relations:

$$s_{ii} = \sum_{j=1}^{3}\left(\frac{\partial S_{ii}}{\partial \Gamma_j} d\Gamma_j\right) \qquad (38)$$

or, using (36) in (38):

$$s_{ii} = \sum_{j=1}^{3}\left[\Gamma_j \frac{\partial S_{ii}}{\partial \Gamma_j} e_{jj}\right] \qquad (39)$$

The general form of the stress-strain relations may be written as

$$s_{ii} = \sum_{j=1}^{3} B_{ij} e_{jj} \qquad (40)$$

Comparing (39) with (40), we have

$$B_{ij} = \Gamma_j \frac{\partial S_{ii}}{\partial \Gamma_j} \qquad (41)$$

Now, for a material that is isotropic in initial finite strain, we assume the relations

$$S_{ii} = \mu\left[\sum_{k=1}^{3} n_{kk}\right] + 2G n_{ii} \qquad (42)$$

Where μ and G are Lame's constants given by

$$\mu = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}$$

and E and ν have the significance of Young's modulus and Poisson's ratio, respectively; n_{ii} are the finite strain components given by

$$2n_{ii} = 1 - \sum_{k=1}^{3} \frac{\partial X^k}{\partial x_i} \frac{\partial X^k}{\partial x_i} \qquad (43)$$

Using (35) in (43) we have

$$n_{ii} = \frac{1}{2}\left(1 - \frac{1}{\Gamma_i^2}\right) \tag{44}$$

We now use (44) in (42) so that the S_{ii} are defined in terms of the Γ_i. The resulting expression is used in (41) to get the moduli B_{ij}. Four of the moduli relevant to plane problems are

$$B_{11} = \frac{\mu + 2G}{\Gamma_1^2} \;;\quad B_{12} = \frac{\mu}{\Gamma_2^2}$$
$$B_{21} = \frac{\mu}{\Gamma_1^2} \;;\quad B_{22} = \frac{\mu + 2G}{\Gamma_2^2} \tag{45}$$

If we invert (42) so that strains n_{ii} are given in terms of stresses S_{ii} and use (44) for the strains, we have

$$\frac{1}{2}\left(1 - \frac{1}{\Gamma_i^2}\right) = \frac{1}{E}\left[S_{ii} - \nu(S_{jj} - S_{kk})\right] \quad (i \neq j \neq k) \tag{46}$$

We now investigate the special case when $S_{22} = S_{33} = 0$ and $S_{11} \neq 0$. In this case, (46) gives

$$\frac{1}{2}\left(1 - \frac{1}{\Gamma_1^2}\right) = \frac{S_{11}}{E}$$
$$\frac{1}{2}\left(1 - \frac{1}{\Gamma_2^2}\right) = \frac{1}{2}\left(1 - \frac{1}{\Gamma_3^2}\right) = -\frac{\nu S_{11}}{E} \tag{47}$$

from which we have

$$1 - 2\frac{S_{11}}{E} = \frac{1}{\Gamma_1^2}$$
$$1 + 2\frac{\nu S_{11}}{E} = \frac{1}{\Gamma_2^2} \tag{48}$$

Substituting (48) in (45) gives

$$B_{11} = (\mu + 2G)\left(1 - 2\frac{S_{11}}{E}\right)$$
$$B_{12} = \mu\left(1 + 2\nu\frac{S_{11}}{E}\right)$$
$$B_{21} = \mu\left(1 - 2\frac{S_{11}}{E}\right) \tag{49}$$
$$B_{22} = (\mu + 2G)\left(1 + 2\nu\frac{S_{11}}{E}\right)$$

The modulus Q was derived by Biot (1963) as

$$Q = \frac{1}{2}(S_{11} - S_{22})\frac{\Gamma_1^2 + \Gamma_2^2}{\Gamma_1^2 - \Gamma_2^2} \tag{50}$$

for materials that are isotropic in finite strain. When S_{11} is the only nonvanishing initial stress, by virtue of (48) we reduce (50) to

$$Q = G\left[1 - \frac{S_{11}}{E}(1 - \nu)\right] \tag{51}$$

Let us set

$$\frac{S_{11}}{E} = -p \tag{52}$$

Then (49) and (50) take the form:

$$B_{11} = (\mu + 2G)(1 + 2p)$$
$$B_{12} = \mu(1 - 2\nu p)$$
$$B_{21} = \mu(1 + 2p) \tag{53}$$
$$B_{22} = (\mu + 2G)(1 - 2\nu p)$$
$$Q = G\left[1 + p(1 - \nu)\right]$$

Note from (47) that if Γ_1 and Γ_2 are the same for both materials A and B, then p is the same for both materials, and also we must have $\nu^A = \nu^B$. Further, note that p represents the finite compressive strain in the x-direction. Equations (53) with proper superscripts to distinguish between A and B are the expressions to be used in the calculations of Section 2.

RESULTS AND DISCUSSION

Equations (53) for the moduli have been used in the results of Section 2 and the various dispersion relations and stability equations have been numerically solved. These equations, as they appear in Section 2, have not been reduced to dimensionless forms so that the symmetry in A and B is apparent. Some manipulation will show, however, that they can be reduced to relations between the following dimensionless quantities:

$$p = -\frac{S_{11}^A}{E^A} = -\frac{S_{11}^B}{E^B} \quad ; \quad \psi = c/\sqrt{G^B/\rho^B}$$

$$\alpha^A = \frac{h^A}{h^A + h^B} \quad ; \quad (\alpha^B = 1 - \alpha^A)$$

$$\zeta = (h^A + h^B)\,a$$

$$\gamma = E^A/E^B$$

$$\theta = \rho^A/\rho^B$$

$$\nu^A \,, \quad \nu^B$$

The results given in this section are in terms of the above parameters. In what follows, Equation $(28)_1$ or $(28)_2$ is to be construed as Equation (28) with K_1 defined by (30) or (31). A similar meaning holds for $(32)_1$ and $(32)_2$.

(a) *Wave propagation*: Figure 3 shows equations (17), (25), $(28)_1$ and $(28)_2$ plotted for certain values of the parameters. The curves show that for infinite wavelength ($a = 0$), all the equations yield the same value of phase velocity. For long wavelengths, the difference between the various theories is not appreciable. For small wavelengths, however, the microstructure theory gives the best approximation to the elasticity solution. This is because the deformation field assumed in the microstructure solution most closely approximates that given by the elasticity solution (see Perkins 1972). A large number of curves were obtained for various values of the parameters, but they have not been presented since Fig. 3 is typical. However, in Figs. 3 a–d, the effect of changing various parameters on the elasticity solution is shown. The other theories give curves that fit relative to these as in Fig. 3, but they have not been presented so as to avoid confusion in the figures.

Sun et al. (1968) used the microstructure approach for the case of zero initial stress, i.e., $\Gamma_1 = 1$; the results presented here for

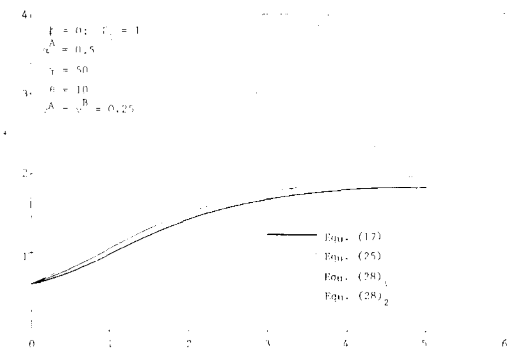

FIG. 3. Phase velocity parameter ψ versus wave number parameter ζ.

$\Gamma_1 = 1$ agree generally with their results. Figure 3d shows the effect of increasing initial compressive stress (decreasing Γ_1) on the dispersion curve. It is seen that increasing the compression decreases the phase velocity for a given wavelength (although for the magnitudes of initial stress considered here, the change is quite small). This agrees with general remarks by Biot (1940, 1965) regarding wave propagation in a continuum under initial stress.

(b) *Elastic stability*: Figures 4 and 5 show Equations (24), (26), $(32)_1$, $(32)_2$ and (34) plotted for various values of the parameters. The elasticity solution for $\alpha^A = 0.1$, $\gamma = 10$ and 50 yields a phenomenon quite distinct from the behavior for other values of the parameters. Attempts to identify these two phenomena with Biot's (1963) "internal instabilities of the second and first kind," respectively, led to no success. Note, however, that the approximate theories do not exhibit the behavior of the elasticity solution for $\alpha^A = 0.1$, $\gamma = 10$ and 50. In the other figures, however, it is again found that the microstructure theory gives the best approximation to the elasticity solution. This is best appreciated in Fig. 4c.

Though Equation (34) has been plotted for the full range of parameters, it is fully realized that it is valid only when the inequalities (33) are satisfied. Note that when such is the case (Fig. 4a, $\gamma = 500$), Equation (34) almost coincides with the elasticity solution. It should also be noted

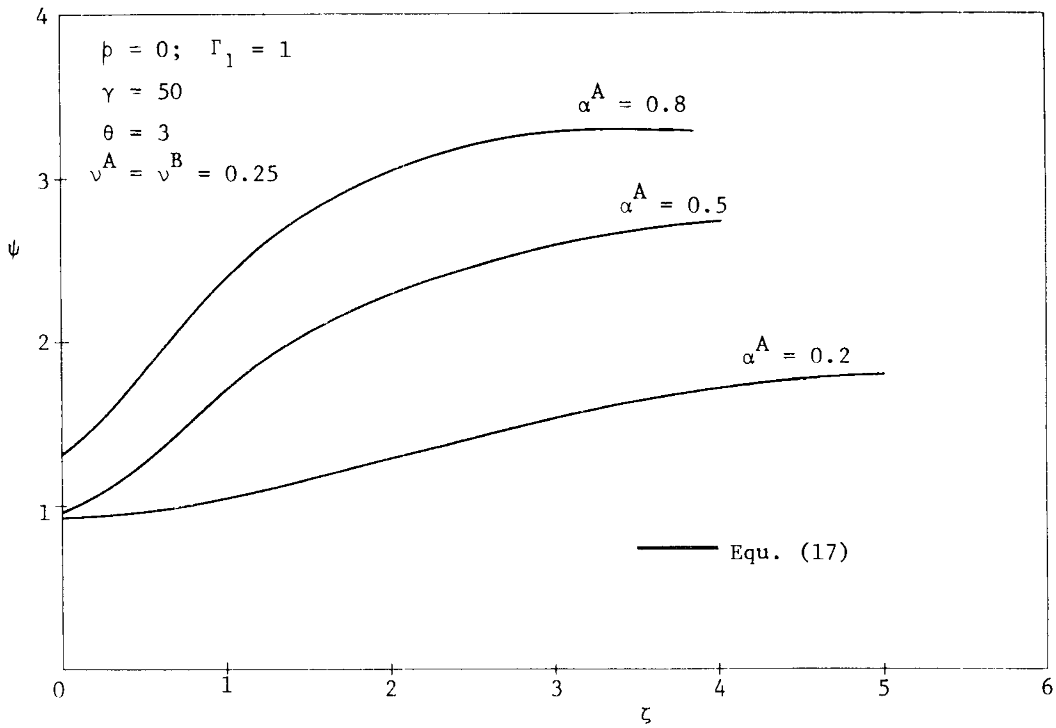

FIG. 3a. Phase velocity parameter ψ versus wave number parameter ζ for various values of layer thickness ratio α^A.

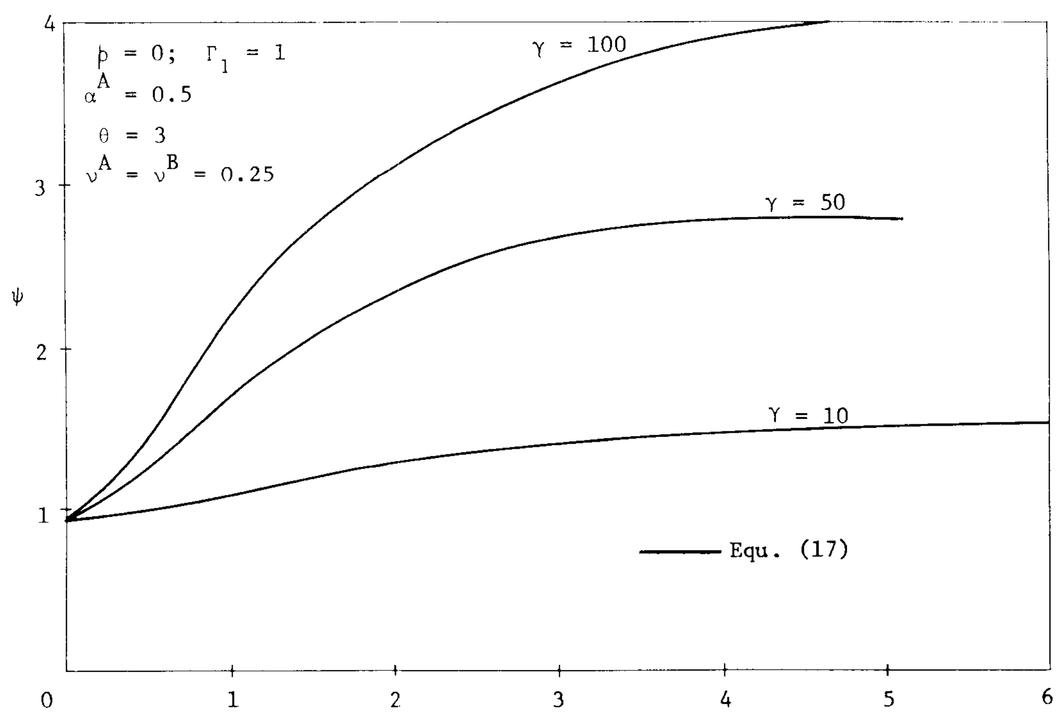

FIG. 3b. Phase velocity parameter ψ versus wave number parameter ζ for various values of the Young's modulus ratio γ.

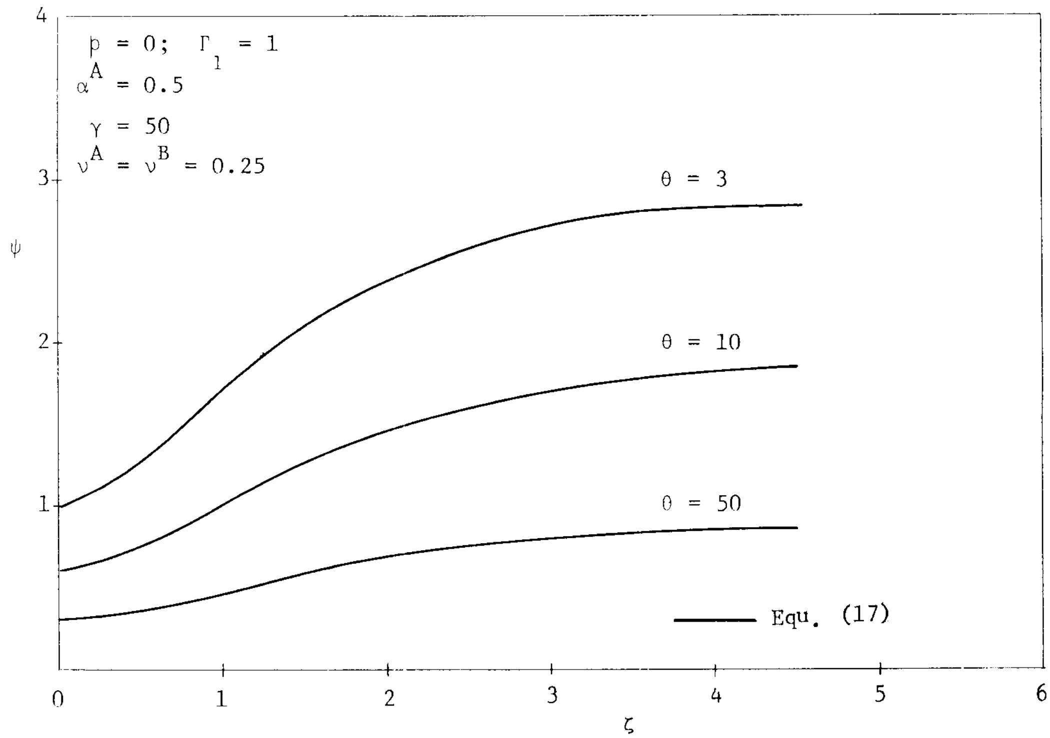

FIG. 3c. Phase velocity parameter ψ versus wave number parameter ζ for various values of the density ratio θ.

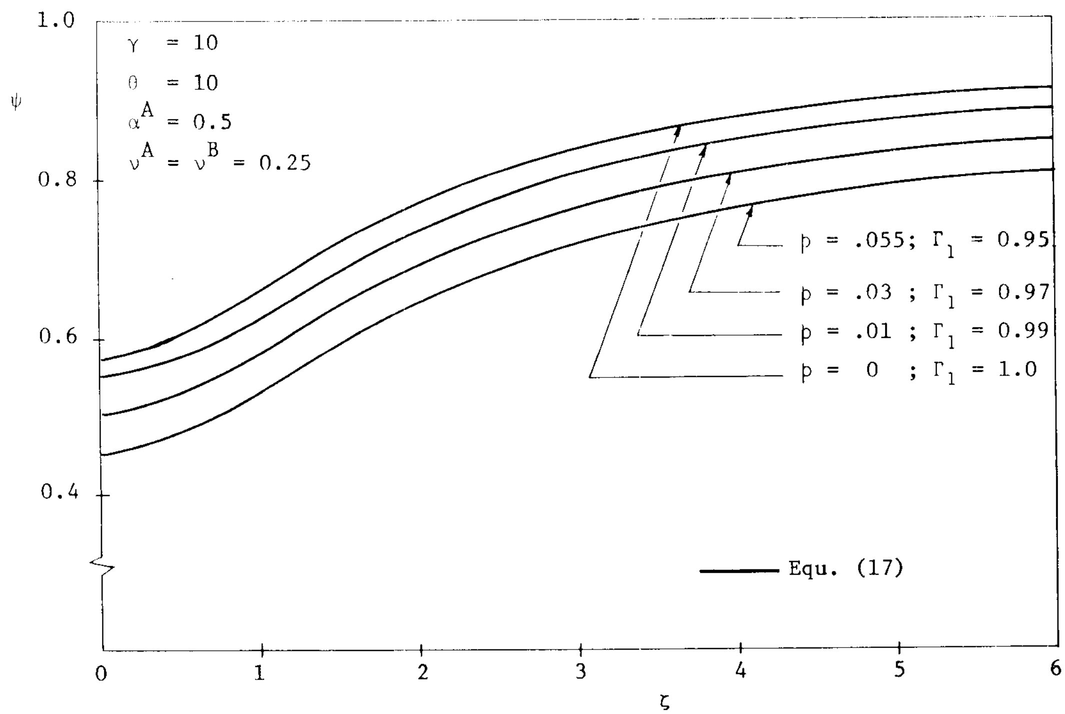

FIG. 3d. Phase velocity parameter ψ versus dimensionless wave number parameter ζ for various values of initial strain.

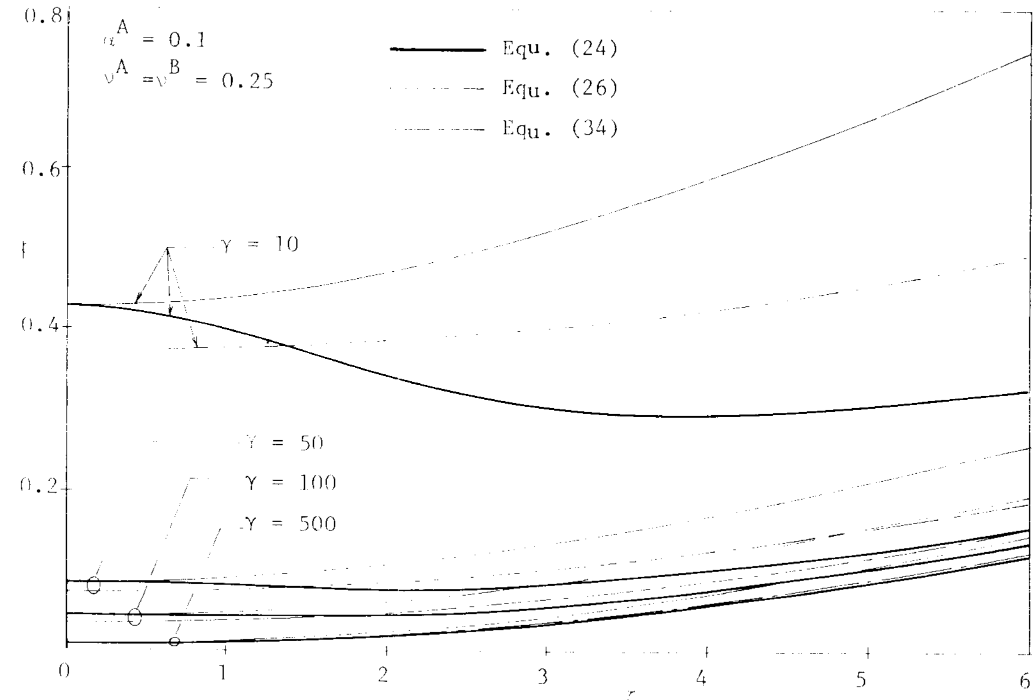

FIG. 4a. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with microstructure solution Eq. (26) and Kiusalaas and Jaunzemis solution Eq. (34). Layer thickness ratio $\alpha^A = 0.1$.

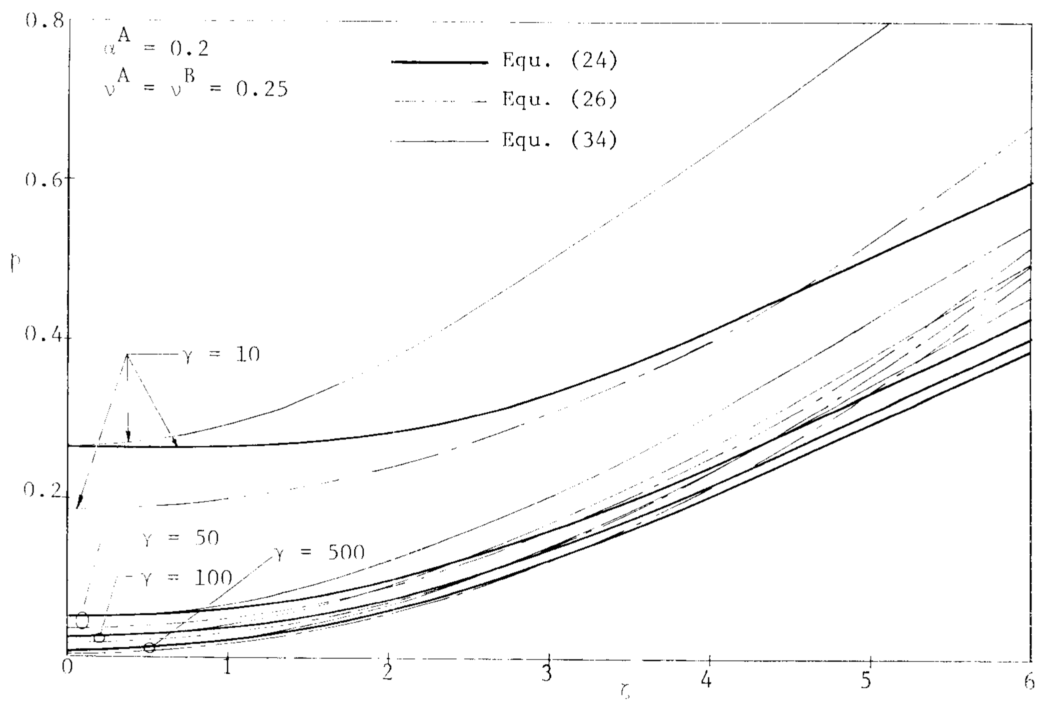

FIG. 4b. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with microstructure solution Eq. (26) and Kiusalaas and Jaunzemis solution Eq. (34). Layer thickness ratio $\alpha^A = 0.2$.

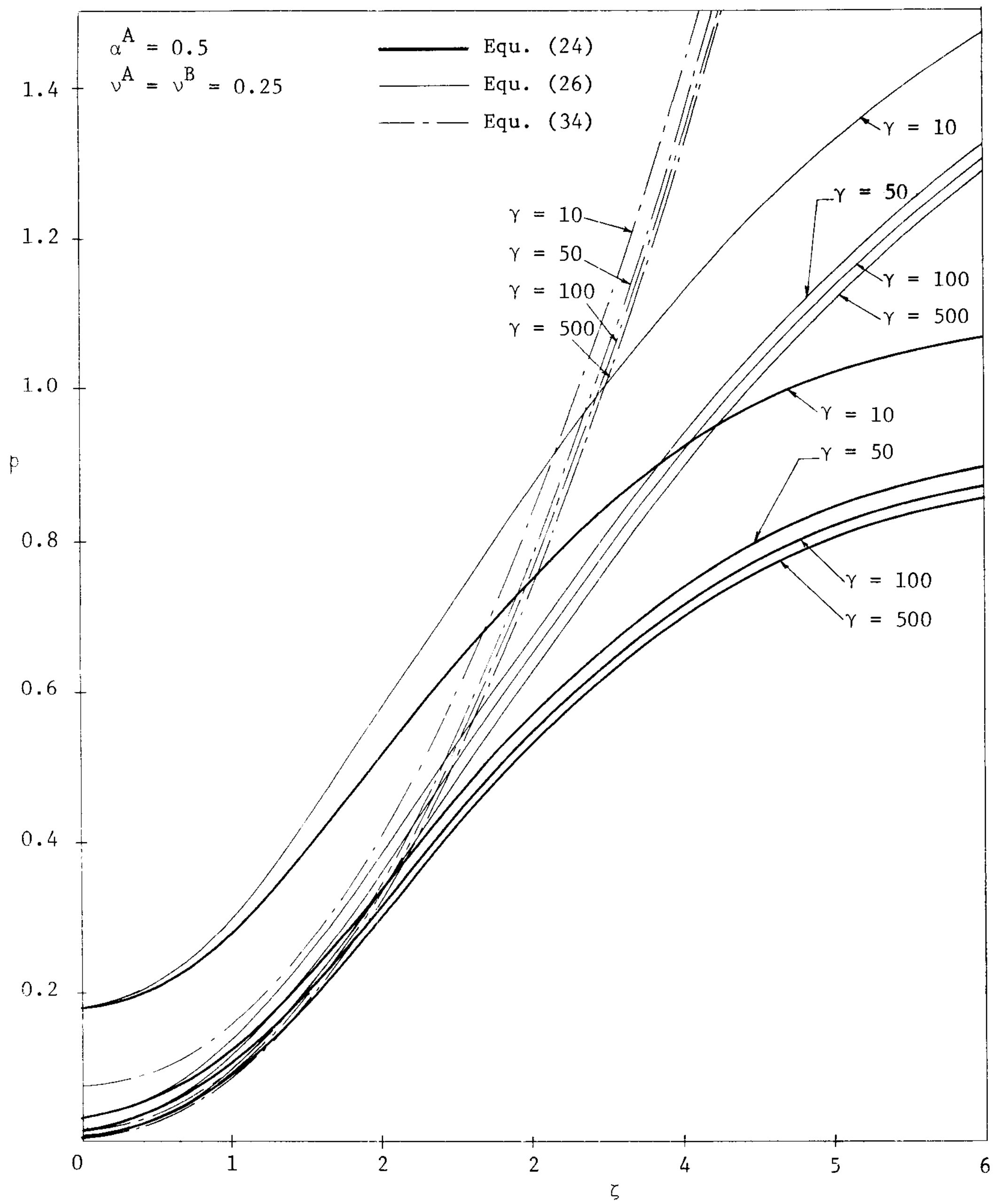

FIG. 4c. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with microstructure solution Eq. (26) and Kiusalaas and Jaunzemis solution Eq. (34). Layer thickness ratio $\alpha^A = 0.5$.

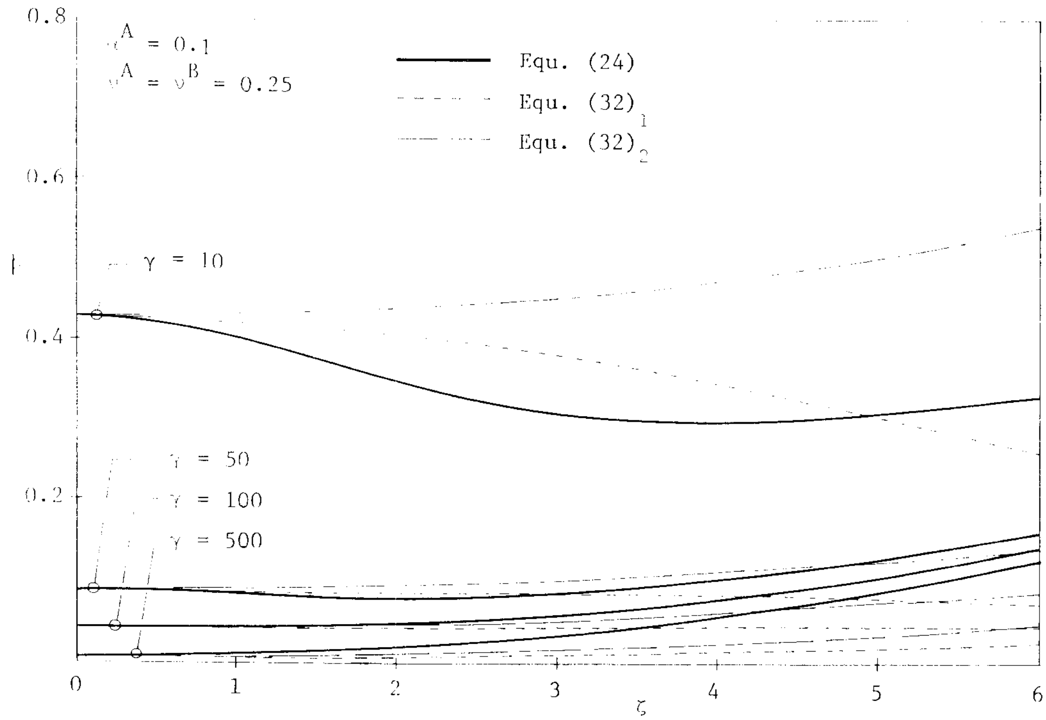

FIG. 5a. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with couple-stress solutions Eq. (32). Layer thickness ratio $\alpha^4 = 0.1$.

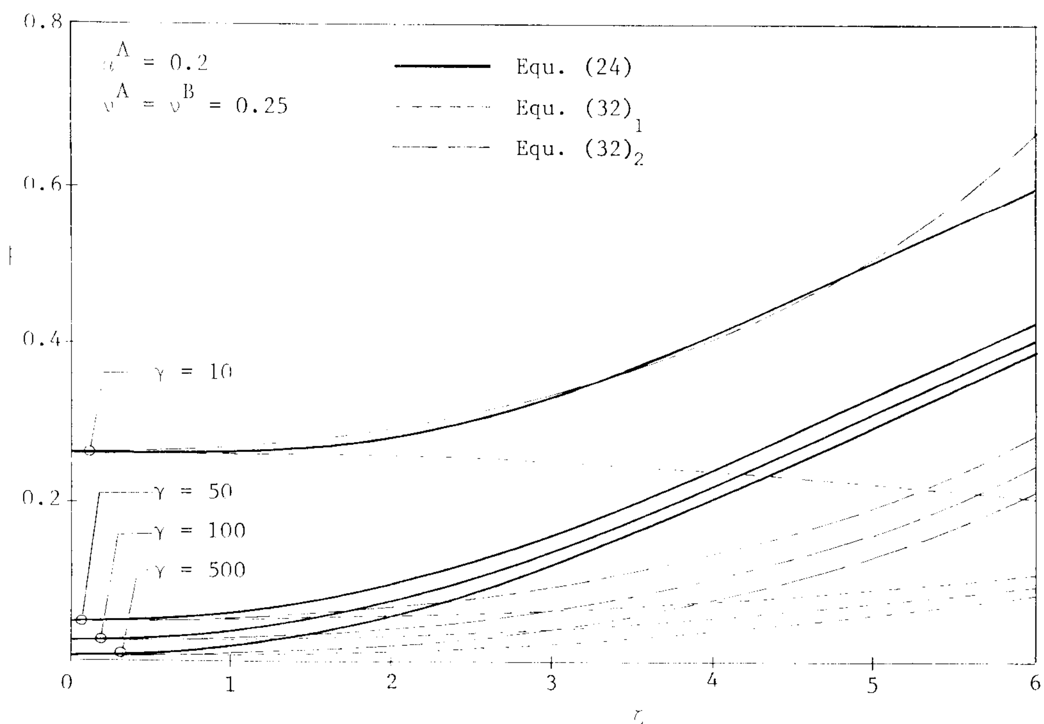

FIG. 5b. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with couple-stress solutions Eq. (32). Layer thickness ratio $\alpha^4 = 0.2$.

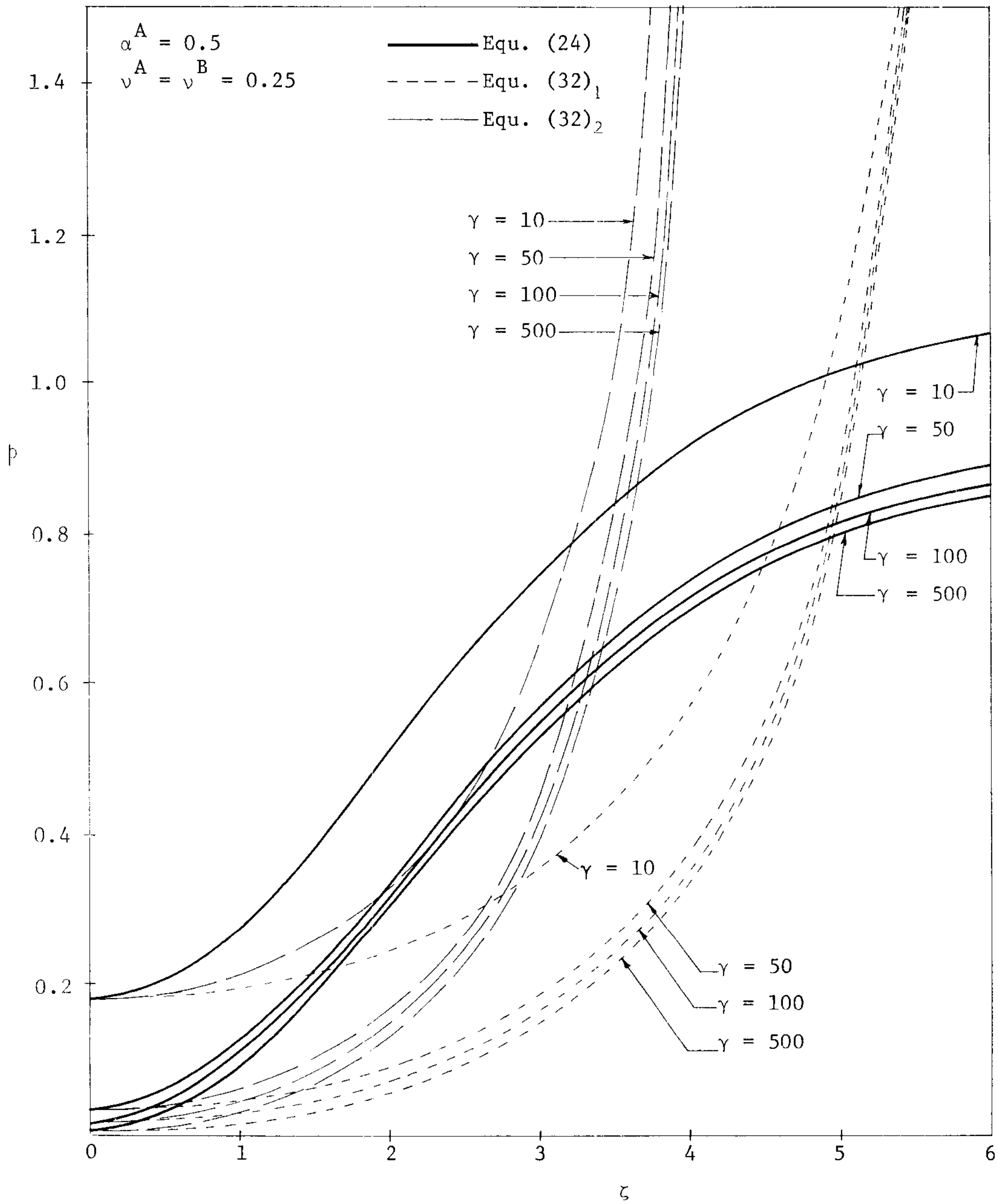

FIG. 5c. Initial stress parameter p versus wave number parameter ζ for various Young's modulus ratios γ. Comparison of exact solution Eq. (24) with couple-stress solutions Eq. (32). Layer thickness ratio $\alpha^A = 0.5$.

that while all other solutions coincide at $a = 0$, Equation (34) deviates considerably when $(\gamma\alpha^4)$ is small. This is probably due to the fact that it is not proper to neglect the prestrain energy in the matrix when $(\gamma\alpha^4)$ is small.

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