STOCHASTIC MODEL FOR MODULUS OF ELASTICITY OF LUMBER

D. E. Kline and *F. E. Woeste*

Graduate Research Assistant and Associate Professor
Agricultural Engineering Department, Virginia Tech
Blacksburg, VA 24061

and

B. A. Bendtsen

Research Forest Products Technologist
Forest Products Laboratory[1]
Madison, WI 53705

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ABSTRACT

A model was developed for generating the lengthwise variability in modulus of elasticity (MOE) of lumber. A limited grade selection of southern pine visual and machine stress-rated (MSR) grades formed the basic data base. A second-order Markov model was used to generate serially correlated MOE's along 30-inch segments for a piece of lumber. Modulus of elasticity indexes were obtained by dividing each correlated MOE by the average MOE of the piece of lumber. The MOE of each segment was obtained by multiplying the MOE indexes by a single random observation from a distribution of MOE. The distribution characteristics of the generated MOE values are preserved, and the first- and second-order lengthwise serial correlations are preserved.

Keywords: Stochastic, modulus of elasticity (MOE), southern pine, lengthwise modeling.

INTRODUCTION

In current design practice, modulus of elasticity (MOE) indicates an average stiffness of the whole piece of lumber. Because a piece of lumber usually contains defect areas such as knots and grain deviations along the piece, presumably a more accurate representation of lumber stiffness would include lengthwise variability in MOE.

Indeed, a number of research studies have identified lengthwise variability in lumber MOE. Corder (1965) measured MOE in 2 × 6 western hemlock at 1-foot intervals along the length using a 2-foot span. Kass (1975) made continuous MOE traces in 2 × 6 southern pine lumber using spans ranging from 8 to 24 inches. Both authors observed lengthwise variability in MOE and that low values of MOE were associated with defects such as knots and slope of grain. Corder noted that the minimum localized MOE value correlated better with bending strength than did full-length MOE, whereas Kass made the same observations for compression strength parallel-to-grain.

Several papers have shown that short-span MOE is a better predictor of tensile

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TABLE 1. *The No. 2 and better KD15 lumber was regraded and assigned to four different groups. The grade bending values F_b and the number of specimens in each group are shown.*

Group	Grade	2 × 4 F_b	2 × 4 No. of specimens	2 × 10 F_b	2 × 10 No. of specimens
		Psi		*Psi*	
1	Select structural dense	2,500	7	2,200	7
2	Select structural	2,150	17	1,850	13
	No. 1 dense	2,150		1,850	
3	No. 1	1,850	18	1,600	20
	No. 2 dense	1,800		1,650	
4	No. 2	1,550	7	1,300	13
Totals			49		53

strength than long-span MOE (Gerhards 1972; Gerhards and Ethington 1974; Orosz 1973, 1976). While none of these papers directly reported lengthwise MOE variability, the better strength correlation for short-span E surely supports its existence and further suggests that the variability is associated with various strength-reducing growth characteristics. These principles are, in fact, the basis for current machine-stress rating (MSR) practices for lumber (Galligan et al. 1977).

Several models have been developed that utilize lengthwise variability in MOE. Suddarth and Woeste (1977) determined the strength of long columns considering the lengthwise variability in MOE. A description of this variability was obtained using observed segmented MOE values measured from four equal intervals along lumber of approximately the same length as the study column.

Structural analysis models using the finite element technique have the capability of assigning a different MOE for each element. For the case of a glued-laminated timber beam, segment values of MOE along the lengths of laminae can be assigned in a finite element model of a laminated beam for a more accurate strength and stiffness prediction (Foschi and Barrett 1980).

The objective of this study was to develop a model that accurately describes lengthwise variability in MOE for several grades and sizes of lumber. It is envisioned that this lumber model can easily be used to develop input for analysis models that utilize lengthwise variability in MOE.

EXPERIMENTAL PROCEDURE

One thousand pieces of 16-foot nominal 2-inch dimension southern pine lumber of two sizes and two grades were obtained on the open market. The numbers of each size and grade of lumber were as follows:

Size	Grade	Number
2 × 4	2250f-1.9E	250
2 × 4	No. 2 KD15	244
2 × 10	2250f-1.9E	250
2 × 10	No. 2 KD15	256

where the No. 2 KD15 is a visual-stress grade denoted (VG) and the 2250f-1.9E is machine-stress rated (MSR).

Approximately 50 specimens from each of the above sizes and grades were

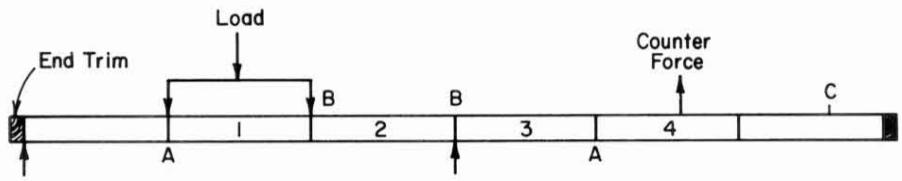

FIG. 1. Location of the four 30-inch segments for flatwise MOE measurements and loading configuration for segment 1 are shown. For segment 2, supports are at points A, load at points B, and counter force at point C. To test segments 3 and 4, the specimen was turned end for end in the equipment. Load was applied perpendicular to the wide face of the specimen.

designated for measurement of MOE. The remaining specimens, approximately 200, were set aside for later experiments to define a length effect for tension strength parallel-to-grain. It was envisioned that the MOE model may be needed as input to the tension strength model, which utilizes a 30-inch length as a base. Based on full-length vibration MOE measurements of all specimens, the 50 specimens that were expected to represent the strength distribution of the entire sample were chosen.

In the southern pine structural lumber market, material grade-stamped No. 2 most commonly includes material No. 2 and better in quality. Therefore, the VG lumber was regraded by a qualified lumber grader into six visual grades and then recombined into four groups based on similarities of allowable bending stress values for the individual grades. Table 1 shows the grades in each group, the allowable bending values, F_b, for the individual grades, and the number of specimens in each group. All lumber was conditioned to an equilibrium moisture content of approximately 12%.

A flatwise static MOE was determined on four 30-inch segments in each specimen. Figure 1 shows the location of each 30-inch segment and the loading configuration. A pre-load and a final load were applied to each test segment with an air-operated ram. The loads were 25 and 100 pounds for the 2 × 4's and 75 and 275 pounds for the 2 × 10's. As the various segments were tested, an upward

TABLE 2. *The number of pieces and lag-1 and lag-2 serial correlation coefficients r_1 and r_2 are shown for each lumber grade and size. N is the number of pieces of lumber.*

Size	Grade	N	r_1	r_2
2 × 4	2250f-1.9E	50	0.778	0.680
2 × 4	Select Structural Dense	7	0.882	0.874
2 × 4	Select Structural No. 1 Dense	17	0.858	0.827
2 × 4	No. 1 No. 2 Dense	18	0.859	0.831
2 × 4	No. 2	7	0.852	0.860
2 × 10	2250f-1.9E	50	0.903	0.834
2 × 10	Select Structural Dense	7	0.838	0.899
2 × 10	Select Structural No. 1 Dense	13	0.882	0.804
2 × 10	No. 1 No. 2 Dense	20	0.904	0.871
2 × 10	No. 2	13	0.921	0.815

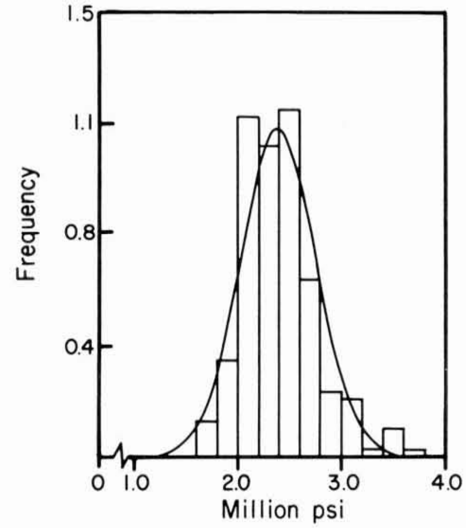

FIG. 2. Two-hundred 30-inch segment MOE values measured from 50 boards were used to form the histogram. A normal distribution fit is superimposed over the histogram.

force was applied to the opposite end to counter the weight of the overhang and to eliminate significant reverse bending moments.

Deflections were measured with an LVDT suspended from the specimen at the load points. This action permitted calculation of a shear-free MOE. The width and thickness, measured to the nearest 0.01 inch at the center of each 30-inch segment, were used to calculate the MOE for individual segments. Three repetitions were performed on each segment, and the MOE values were obtained from the average of the three measurements. Repeatability for the three measurements was generally within 1–3%.

RESULTS

Serial correlation of MOE

The correlation structure of MOE was considered along the piece of lumber. It is reasonable to assume that lumber exhibits serial correlation; i.e., that the MOE in one segment is correlated to the MOE in the previous segment. Lag-K serial correlation ρ_K is the correlation between an observation at one interval length and an observation at K previous intervals. For example, if lag-2 serial correlation exists, there is a significant correlation between the MOE of the first and third 30-inch segments and the second and fourth 30-inch segments of a 120-inch piece of lumber. Table 2 shows the estimated lag-1 and lag-2 serial correlation coefficients r_1 and r_2 for each grade and size. All correlations were high, ranging from 0.68 to 0.92. If N is the number of pieces of lumber, then r_1 was calculated with 3*N observations and r_2 was calculated with 2*N observations.

The purpose of calculating r_1 and r_2 for the regraded VG lumber was to determine if any glaring differences in the statistics existed for the different grades. There are no glaring differences in the coefficients in Table 2 especially when considering the sample sizes involved. We cannot determine from this experiment if a difference exists in serial correlation of MOE for VG southern pine lumber. We

TABLE 3. *The number of pieces and lag-1 and lag-2 serial correlation coefficients r_1 and r_2 are shown for each lumber grade and size. N is the number of pieces of lumber.*

Size	Grade	N	r_1	r_2
2 × 4	2250f-1.9E	50	0.778	0.680
2 × 4	No. 2[1]	49	0.908	0.882
2 × 10	2250f-1.9E	50	0.903	0.834
2 × 10	No. 2[1]	53	0.924	0.871

[1] The No. 2 lumber resulted from a combination of the four regraded groups.

decided, for purposes of demonstrating the method of modeling, to recombine the visual grades.

Distribution of segment MOE

A Markov normal process was anticipated as the underlying model; thus histograms of the 30-inch segment MOE's were prepared for each of the two sizes and two grades. A normal distribution was fitted to the 30-inch MOE values. There were four times as many MOE observations as specimens since four segments were measured on each board. Each histogram thus was formed with MOE values for approximately 200 segments. The fitted density curves and histograms were visually inspected for conformance of the normal fit to the data. Of the four lumber groups, Fig. 2 for the MSR 2 × 10 data shows the greatest lack of fit in the tail regions. Because these MOE data are formed by groups of four correlated observations, standard statistical goodness-of-fit tests are not applicable. Since three of the four fits were excellent, and one fit was assessed as fair, we accepted the normal distribution fits. Parenthetically, the segment MOE's are not used in design but only in an indexing procedure for making lengthwise MOE adjustments to prescribed MOE distributions.

The lag-1 and lag-2 serial correlation coefficients were calculated for the MSR and combined VG lumber (grade-stamped No. 2 KD15) of each size (Table 3). The increased sample size that resulted from combining the regraded lumber improved estimates for ρ_1 and ρ_2. The four lumber grades and sizes listed in Table 3 were used to build four models.

A Markov model was used to model the serial correlation structure observed. The lag-m or mth order Markov model is given by Haan (1977) as

$$X_{i+1} = \beta_o + \beta_1 X_i + \beta_2 X_{i-1} + \ldots + \beta_m X_{i-m+1} + \epsilon_{i+1} \qquad (1)$$

TABLE 4. *The parameter estimates for the first-order Markov model are shown for each grade and size of lumber. N is the number of pieces of lumber.*

Size	Grade	N	β_1	R^2	σ_X[1]	μ_X[1]
					× 10^5 psi	*× 10^6 psi*
2 × 4	2250f-1.9E	50	0.778	0.605	3.775	2.571
2 × 4	No. 2[2]	49	0.907	0.824	5.075	1.749
2 × 10	2250f-1.9E	50	0.903	0.816	3.601	2.402
2 × 10	No. 2[2]	53	0.924	0.854	6.006	1.770

[1] σ_X and μ_X are the standard deviation and mean of the 30-inch MOE segments.
[2] The No. 2 lumber resulted from a combination of the four regraded groups.

TABLE 5. *The parameter estimates for the second-order Markov model are shown for each grade and size of lumber. N is the number of pieces of lumber.*

Size	Grade	N	β_1	β_2	R^2	σ_X[1]	μ_X[1]
						$\times 10^5$ *psi*	$\times 10^6$ *psi*
2 × 4	2250f-1.9E	50	0.630	0.190	0.619	3.775	2.571
2 × 4	No. 2[2]	49	0.606	0.332	0.843	5.075	1.749
2 × 10	2250f-1.9E	50	0.817	0.095	0.818	3.601	2.402
2 × 10	No. 2[2]	53	0.816	0.117	0.856	6.006	1.770

[1] σ_X and μ_X are the standard deviation and mean of the 30-inch MOE segments.
[2] The No. 2 lumber resulted from a combination of the four regraded groups.

where the X_i's represent the observed data values and the β's are multiple regression coefficients. If normality in the data is assumed, the random element becomes

$$\epsilon_{i+1} = \sigma_X t\sqrt{1 - R^2} \tag{2}$$

where $\sigma_X{}^2$ is the variance of X, R^2 is the coefficient of determination resulting from Eq. 1, and t is a random observation from a standard normal distribution, N (0, 1).

The first- and second-order Markov models were fitted to the MOE data and analyzed for the best fit. The first term in Eq. 1, β_o, is reduced to zero if X is constructed so that its expected value is zero, and thus the first-order Markov process is simplified by

$$X_{i+1} = \beta_1 X_i + \epsilon_{i+1} \tag{3}$$

where the mean of X, μ_X, is zero.

The lag-1 serial correlation is preserved when $\beta_1 = \rho_1$ (Haan 1977). The first-order Markov model generates serial correlations of any lag-K by the theoretical model

$$\rho_K = \rho_1{}^K \tag{4}$$

where ρ_1 is estimated by r_1.

The second-order Markov model is given by

$$X_{i+1} = \beta_1 X_i + \beta_2 X_{i-1} + \sigma_{i+1} \tag{5}$$

The lag-1 and lag-2 serial correlations are both preserved with the second-order Markov model when

$$\beta_1 = (\rho_1 - \rho_1\rho_2)/(1 - \rho_1{}^2) \tag{6}$$

and

TABLE 6. *The observed lag-3 serial correlation coefficients r_3 are compared to the theoretical values of the first- and second-order models for each grade and size of lumber.*

			Theoretical ρ_3	
Size	Grade	r_3	First-order	Second-order
2 × 4	2250f-1.9E	0.610	0.471	0.576
2 × 4	No. 2[1]	0.919	0.747	0.836
2 × 10	2250f-1.9E	0.699	0.737	0.767
2 × 10	No. 2[1]	0.805	0.789	0.819

[1] The No. 2 lumber resulted from a combination of the four regraded groups.

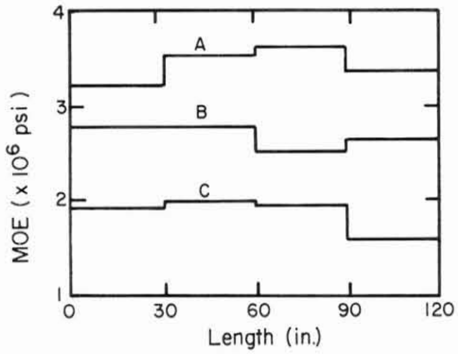

FIG. 3. The lengthwise variability in MOE as measured on a 30-inch span is shown for three specimens of the 2 × 4 MSR lumber.

$$\beta_2 = (\rho_2 - \rho_1^2)/(1 - \rho_1^2) \tag{7}$$

where ρ_1 and ρ_2 are estimated by r_1 and r_2, respectively (Yevjevich 1972). The second-order Markov model generates serial correlations according to the theoretical model

$$\rho_K = \beta_1\rho_{K-1} + \beta_2\rho_{K-2} \tag{8}$$

The mean μ_X and standard deviation σ_X of segment MOE values were calculated for each of the four grade-size combinations considered in Table 3. Each grade was then standardized by subtracting the grand mean of all segments from each individual segment MOE value. This standardization results in a variable that has a mean of zero, which then enables the use of the Markov models of Eqs. 3 and 5. Tables 4 and 5 show the estimated parameters required for the first- and second-order Markov models, respectively. β_1 in Table 4 was set equal to r_1, and β_1 and β_2 in Table 5 were calculated with Eqs. 6 and 7.

β_1 and β_2 in Eq. 5 were also estimated by multiple linear regression. F-tests were performed to test the hypothesis that β_2 was significantly different from zero. All estimates of β_2 were not statistically significant with α equal to 0.01. The F-test indicated that a second-order Markov model did not significantly explain any

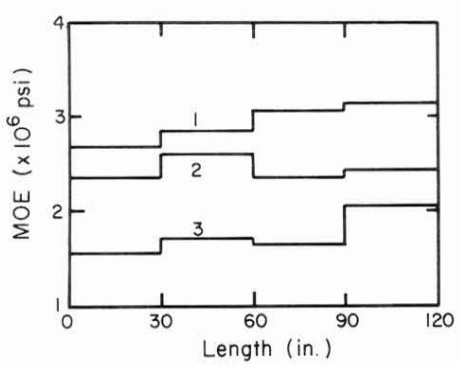

FIG. 4. The lengthwise 30-inch segment variability in MOE is shown for three generated pieces of lumber. The MOE traces for specimens 1, 2, and 3 are from a normal process.

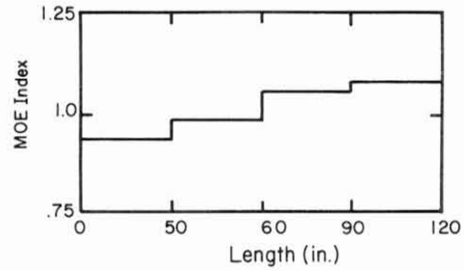

FIG. 5. The MOE indexes for generated specimen 1 are shown. The MOE indexes were obtained by dividing individual MOE's of specimen 1 by the specimen average.

more of the variability of the MOE data than did the first-order Markov model. However, the second-order Markov model preserved both lag-1 and lag-2 serial correlations of the MOE data, whereas the first-order Markov model preserves only the lag-1 serial correlation. Furthermore, the theoretical lag-3 serial correlation of the second-order Markov model, as calculated by Eq. 8, was closer to the observed lag-3 serial correlation than was the alternative. The highest lag serial correlation possible that could be observed was for lag-3 because the MOE data were measured at only four 30-inch intervals for each specimen. Table 6 compares the observed lag-3 serial correlation of the MOE data to the theoretical values predicted by the first- and second-order Markov models. In three of four cases, the 2 × 10 MSR lumber excepted, the second-order model predicted a lag-3 serial correlation closer to the observed lag-3 serial correlation coefficient than did the first-order model. Therefore, the second-order Markov model was chosen as the more appropriate model.

A third-order Markov model was not considered because of its statistical complexity and because the second-order model appeared to be adequate.

Generation of MOE

A model was developed that generated a different MOE value every 30 inches along the length of a piece of lumber. Figure 3 shows the variability in MOE for three observed specimens of the 2 × 4 MSR lumber. Each of the connected lines A, B, and C represents the variability in MOE along the length of each specimen in 30-inch segments. The averages of lines A, B, and C each have their own random variability, in this case described by a 3-parameter Weibull distribution.

In summary, two processes are occurring: (1) MOE varies within each piece of lumber, and (2) the average MOE of each piece of lumber has a variability that is described by some theoretical MOE distribution. The 30-inch segment MOE variability within each piece of lumber was shown to be modeled by a second-order Markov process.

To start the generation process, X_i and X_{i-1} in Eq. 5 were arbitrarily set equal to zero and 10 values were generated and discarded. This action is required to eliminate bias in the first MOE segment generated. The 11th value was assigned to the first 30-inch segment, the 12th value to the second 30-inch segment, and so on for each generated specimen. Because the second-order Markov model generates MOE deviates, the mean value of the observed MOE was added to each segment MOE deviate.

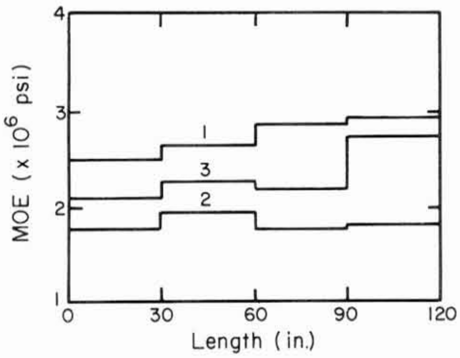

FIG. 6. The MOE traces are shown for the three generated specimens. The average MOE of the segments for each piece follows any prescribed probability distribution.

Figure 4 shows three generated specimens of 2 × 4 MSR lumber. The average of all possible lines, such as those depicted in Fig. 4, should have a mean of 2.571 million psi and a standard deviation of 0.3775 million (Table 5). The MOE traces of Fig. 4 are an intermediate step in the MOE generation process. The traces—the result of adding the mean of 2.571 million psi to Eq. 2—are not directly useful unless one cares to generate normally distributed segment MOE data with mean 2.571 million psi and standard deviation 0.3775 million psi. For MOE generation from any other distribution, the indexing procedure described in the next paragraph must be used.

To preserve the distribution of the average MOE and correlation structure of each piece of generated lumber, MOE indexes were obtained. The average-piece MOE distribution must be prescribed along with its theoretical coefficient of variation. The standard deviation, σ_X, in Eq. 2 must be changed to the product of the coefficient of variation of the prescribed distribution and the observed segment mean, μ_X, from Table 5. Because of the high MOE serial correlation in lumber, the coefficient of variation of the MOE segments can be approximated by the coefficient of variation of the apparent MOE of full-size lumber beams. This approximation was verified as being accurate for the case of a beam 120 inches long under two equal loads at the quarter points. Each of the four lumber quality classes of Table 5 was studied using Monte Carlo simulation and the conjugate beam method of analysis wherein the MOE of each 30-inch segment was varied in accordance with the second-order Markov model. In no case did the average of the MOE's of the beams differ by more than 0.63% from the average MOE of the segments. The maximum difference for the coefficients of variation of MOE was 0.136 versus 0.147, the latter calculated for the segments. This difference is not of practical significance.

Modulus of elasticity indexes were calculated by dividing each of the generated-segment MOE values by the mean of the segments for one piece. The MOE indexes in Fig. 5 were obtained from the generated specimen No. 1 of Fig. 4. The indexes for a single piece of lumber will always vary about a mean value of 1.0, as was the case for the example specimen of Fig. 5. Three random observations from the prescribed MOE distribution were generated and multiplied by three sets of MOE indexes. Figure 6 shows MOE traces for three generated specimens. The generated MOE traces in Fig. 6 have the same lag-1 and lag-2 serial correlations as do the observed MOE data in Fig. 3.

SUMMARY

This study shows that a second-order Markov process models the lengthwise variability in MOE of two sizes and two grades of southern pine lumber samples. The required parameters for the second-order Markov model are summarized in Table 5. Using any prescribed MOE distribution, lengthwise segment MOE values having the same statistical characteristics as the developmental sample can be generated. A summary of the generation procedure for one piece of lumber is as follows:

1. Using a second-order Markov model, a specified number of serially correlated MOE 30-inch segment values are generated.
2. The average of the segment MOE's is calculated.
3. Modulus of elasticity indexes are obtained by dividing each segment MOE by the piece-average MOE.
4. A random-piece MOE is generated from a prescribed probability distribution of the desired size and grade of lumber.
5. Lengthwise segment MOE values are obtained by multiplying the random observation of MOE by the MOE indexes.

CONCLUSIONS

A model was developed that generates 30-inch lengthwise variability in MOE for two sizes and two grades of southern pine lumber. The MOE of the generated specimens has any distribution as prescribed. The generated lengthwise MOE segments have one of four correlation structures that are associated with the lumber grades and sizes sampled. The MOE can be generated for 30-inch segments of a piece of lumber of any length. Lengthwise variability in MOE, instead of piece averages, can easily be used as input to strength and stiffness structural models for research and design.

Further research in this area is needed to define possible size and species effects as well as the effect of grading method and different segment lengths. The development of models for segment lengths less than 30 inches should have the highest priority.

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