A THEORY OF MECHANICAL RESPONSE OF FIBER NETWORKS[1]

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(Received 30 December 1971)

ABSTRACT

General theoretical aspects of a continuum mathematical model for predicting the mechanical behavior of fiber networks are presented. The theory is developed for an idealized two-dimensional elastic network subjected to static loading. It is intended that the theory be employed to study the elastic stress-strain behavior of paper sheets or other bonded fibrous materials.

Additional keywords: Mathematical models, continuum theory, paper elasticity, paper, nonwovens, analysis.

INTRODUCTION

It is the purpose of this work to describe the general theoretical aspects of a continuum mathematical model for predicting the mechanical response of fiber networks that are subjected to external loading as well as thermal and moisture expansion. The work is devoted primarily to the theoretical development of an ideal plane network model that behaves elastically and that is assumed to be in a state of static equilibrium.

It is recognized that the ideal two-dimensional network model is not adequate quantitatively to predict every aspect of the mechanical behavior of paper sheets; however, it serves to provide a foundation for the development of such a theory. From the theoretical point of view, it is not particularly difficult to extend the present work to incorporate dynamic effects and nonelastic behavior as well as certain aspects of nonplanar response of the fiber elements. These features have purposely been postponed for the present in the interest of emphasizing the critical aspects of the proposed theory that are essential to the development of a rational theory of the fiber network.

The predicted mechanical behavior of a paper sheet can be synthesized from a model system consisting of an essentially two-dimensional network of fiber elements. The fiber elements are taken to be portions of wood fibers located between bonds in the network.

In order to describe realistically the properties of the network, it is necessary to provide the structural elements, i.e., the fiber elements and bonds, with degrees of freedom of their own that may differ locally from those of the average response of the network. At any arbitrary point in the network, the fiber segment located there may deform much differently from that which one would anticipate from a knowledge of the gross strain and displacement fields of the network. For example, consider a fiber segment oriented in the direction of a uniaxially applied tensile stress for an isotropic network. The network can be expected to elongate in the direction of applied stress and contract in the direction normal to the direction of applied stress. The above-mentioned fiber segment, however, may elongate or not, or it may deform by bending. Local deviations in deformation from those of the gross network deformation are caused by the nonhomogeneity of the network system. Accounting for this aspect of the system behavior is an essential part of the solution of the network problem.

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[1] Research reported in this paper was supported by the National Science Foundation under grant GK-3810.

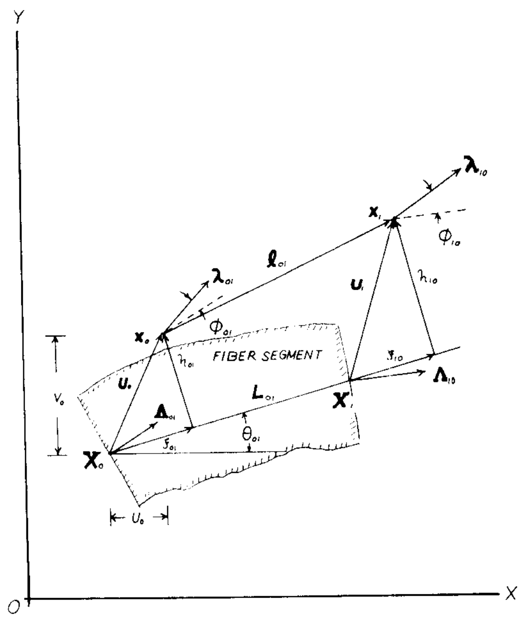

FIG. 1. Kinematic measures for a typical fiber segment located between bonds $\mathbf{X}_0$ and $\mathbf{X}_1$.

From the mathematical modeling point of view, it is greatly advantageous to be able to describe the network behavior with a continuum model. If one were to attempt to synthesize sheet behavior from a model consisting of a discrete number of fiber elements, an immensely difficult problem arises because the number of equations that must be solved simultaneously is proportional to the number of discrete elements. It is obvious that such an approach would become computationally unmanageable for any realistic model of a paper sheet. In the following, it is shown how a continuum theory with local degrees of freedom can be employed to model the elastic behavior of a network model system. The presently described theory is an application of certain aspects of several so-called microstructural continuum theories that have recently been proposed for predicting the mechanical behavior of nonhomogeneous media. See, for example, Klemm and Wózniak (1970), Mindlin (1964), Wózniak (1967), Eringen and Sukubi (1964).

THEORY OF AN IDEAL TWO-DIMENSIONAL NETWORK

General assumptions

The network is comprised of a set of fiber segments, the ends of which are joined or "bonded" to each other. The fiber segments are presumed to be in the form of flattened strips. The fiber segment material is assumed in general to be an anisotropic elastic material. The segments are presumed to transmit a resultant axial (compressive or tensile) load, and as well, shear and bending moment resultants. In general typical fiber segments may be assumed to experience axial shortening or elongation, shear, and bending deformation. The bond material associated with the joining of fiber segments is assumed also to behave elastically and therefore to deform as a result of the loads transmitted through the bonds by the fiber segments.

Kinematic description

The geometrical and kinematic measures of network constitution and deformation are illustrated in Fig. 1. The plane of the network is represented by the xy-plane. The locations of bonds in the undeformed network are provided by the vectors $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, ... The bonds are located relative to one another by the set of vectors $\mathbf{L}_{01}$, $\mathbf{L}_0$, . . . , $\mathbf{L}_{12}$, $\mathbf{L}_{13}$, ... The vectors $\mathbf{L}_{01}$, $\mathbf{L}_{02}$, . . . have orientations θ_{01}, θ_{02}, . . . with the positive x-axis. The centerline of the fiber segment joining bond $\mathbf{X}_0$ with bond $\mathbf{X}_1$ is denoted by the Λ_{01}, etc. As a result of loading and deformation of the network, the bonds originally located at $\mathbf{X}_0$, $\mathbf{X}_1$, . . . move to locations $\mathbf{x}_0$, $\mathbf{x}_1$, . . . , the relative bond position vectors $\mathbf{L}_{01}$, $\mathbf{L}_{02}$, . . . become $\boldsymbol{l}_{01}$, $\boldsymbol{l}_{02}$, . . . and the fiber segment end orientation unit vectors $\boldsymbol{\Lambda}_{01}$, $\boldsymbol{\Lambda}_{02}$, . . . become unit vectors $\boldsymbol{\lambda}_{01}$, $\boldsymbol{\lambda}_{02}$, ... The ends of the fiber segment located between bonds $\mathbf{X}_0$ and $\mathbf{X}_1$ rotate by the amounts $\phi_{01} = \text{arc cos } \boldsymbol{\Lambda}_{01} \cdot \boldsymbol{\lambda}_{01}$, $\phi_{10} = \text{arc cos } \boldsymbol{\Lambda}_{10} \cdot \boldsymbol{\lambda}_{10}$. The bond displacement $\mathbf{u}_0$ of the $\mathbf{X}_0$ bond can be represented in terms of rectangular components u_0, v_0 in the x, y – directions or in terms of the ξ_{01}, η_{01} components measured along and perpendicular to a line joining bonds $\mathbf{X}_0$, $\mathbf{X}_1$.

Strain energy of a finite network

The fiber segment joining bonds $\mathbf{X}_0$ and $\mathbf{X}_1$ experiences an axial deformation $\epsilon_{01} = \xi_{10} - \xi_{01}$, a lateral displacement of one end relative to the other end $\delta_{01} = \eta_{10} - \eta_{01}$, and rotations of the ends ϕ_{01} and ϕ_{10}. It is assumed that the fiber segment transmits a resultant axial load N_{01}, a resultant shear load V_{01}, and bending moment resultants M_{01} and M_{10}. The relationships between loading and deformation of the fiber segment between $\mathbf{X}_0$ and $\mathbf{X}_1$ are assumed to be of the form

$$\begin{aligned} N_{01} &= K_{11}\epsilon_{01} - B_1\Delta T \\ V_{01} &= K_{22}\delta_{01} + K_{23}\phi_{01} + K_{24}\phi_{10} - B_2\Delta T \\ M_{01} &= K_{23}\delta_{01} + K_{33}\phi_{01} + K_{34}\phi_{10} - B_3\Delta T \\ M_{10} &= K_{24}\delta_{01} + K_{34}\phi_{01} + K_{44}\phi_{10} - B_4\Delta T\ , \end{aligned} \tag{1}$$

where the coefficients K_{11}, K_{22}, K_{33}, ... depend upon the material properties, dimensions, and curvature of the fiber segment. In general, the coefficients are dependent upon the distance between bonds L_{01}, the orientation of the segment θ_{01}, and the location $\mathbf{X}_0$ of the fiber segment reference bond in the plane of the network. The coefficients B_1, B_2, B_3, B_4 represent the thermal expansion or moisture shrinkage effect, while ΔT represents either a change in temperature or a change in moisture content from some reference value.

The elastic energy $W_{ij}{}^F$ stored in a fiber segment between bonds $\mathbf{X}_i$ and $\mathbf{X}_j$ can be expressed as

$$\begin{aligned} W_{ij}^F &= \frac{1}{2}K_{11}\epsilon_{ij}^2 + \frac{1}{2}K_{22}\delta_{ij}^2 + \\ &\frac{1}{2}K_{33}\phi_{ij}^2 + \frac{1}{2}K_{44}\phi_{ji}^2 + K_{23}\phi_{ij}\delta_{ij} + \\ &K_{24}\phi_{ji}\delta_{ij} + K_{34}\phi_{ij}\phi_{ji} - B_1\Delta T\epsilon_{ij} - \\ &B_2\Delta T\delta_{ij} - B_3\Delta T\phi_{ij} - B_4\Delta T\phi_{ji}\ . \end{aligned} \tag{2}$$

The elastic energy associated with deformation of the bond at $\mathbf{X}_i$, $W_i{}^B$ can be written as

$$W_i^B = \sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\frac{1}{4}\ {}_iK_{jk}^B\left[\phi_{ik} - \phi_{ij}\right]^2 \tag{3}$$

where N_i represents the number of bonds neighboring bond $\mathbf{X}_i$, and ${}_iK_{jk}{}^B$ represents the moment per unit bond deformation measured by the difference $(\phi_{ik} - \phi_{ij})$ at bond $\mathbf{X}_i$ oriented between the fiber segments extending to bond $\mathbf{X}_j$ and bond $\mathbf{X}_k$. Then, if

$$W_i{}^F = \sum_{j=1}^{N_i} W_{ij}^F \tag{4}$$

represents the strain energy stored in the fiber segments emanating from bond $\mathbf{X}_i$ and if M represents the total number of bonds in the network, the total elastic energy stored in the deformed network, U, is

$$U = \frac{1}{2}\sum_{i=1}^{M} W_i^F + \sum_{i=1}^{M} W_i^B \tag{5}$$

Connections between microscopic and macroscopic network deformation

If the network consists of a finite number of bonds M and fiber segments joining the bonds, the total energy stored in the system can be calculated for a given distribution of bonds $\mathbf{X}_0$, $\mathbf{X}_1$, ... Since the number of elements in a practical situation is very large, it is expedient to attempt to describe the deformation of the network as though it were a continuum. In order to accomplish this, it is necessary to establish a connection between the deformation of individual fiber segments (the microscopic level) and the continuum description of the deformation of the network (the macroscopic level). This may be carried out in a variety of different ways. Three possible connections are proposed in this paper.

Let us refer to the microscopic displacements of a typical fiber segment having ends initially at $\mathbf{X}_0$, $\mathbf{X}_1$ by the quantities

u_0', v_0', and u_1', v_1'. The macroscopic displacements of the continuum at point $\mathbf{X}_0$ we assume are given by the functions u(x,y), v(x,y). The axial deformation of the fiber segment is given by the expression

$$\epsilon_{01} = m\,(u_1' - u_0') + n\,(v_1' - v_0'), \quad (6)$$

where $m = \cos\theta_{01}$, $n = \sin\theta_{01}$ and the relative deflection is given by the expression

$$\delta_{01} = -n\,(u_1' - u_0') + m\,(v_1' - v_0'). \quad (7)$$

In general, we propose that the microdisplacements u_1', v_1', u_0', v_0' are functions of the continuum displacements u, v at $\mathbf{X}_0$ and various gradients of the displacement functions. Thus, we write[2]

$$\begin{aligned} u_0' &= f_0(u, v, u_x, v_x, \ldots), \\ u_1' &= f_1(u, v, u_x, v_x, \ldots), \\ v_0' &= g_0(u, v, u_x, v_x, \ldots), \\ v_1' &= g_1(u, v, u_x, v_x, \ldots). \end{aligned} \quad (8)$$

Likewise, the microscopic measures of rotation of the ends of the fiber segment with ends at $\mathbf{X}_0$ and $\mathbf{X}_1$ are denoted ϕ_{01}' and ϕ_{10}'. If $\phi(\theta$,x,y) represents a continuum rotation function, which generally depends on orientation θ and position x,y, we write

$$\begin{aligned} \phi_{01}' &= h_0(\phi, \phi_x, \phi_y, \phi_\theta, \ldots), \\ \phi_{10}' &= h_1(\phi, \phi_x, \phi_y, \phi_\theta, \ldots). \end{aligned} \quad (9)$$

CONNECTION METHOD I

The most straightforward and the easiest computational method for providing a connection between the macroscopic and microscopic levels is to pick

$$\begin{aligned} & f_0 = u, \quad g_0 = v, \quad f_1 = u + u_x L m + u_y L n \\ & g_1 = v + v_x L m + v_y L n, \\ & h_0 = \phi, \quad h_1 = \phi + \phi_x L m + \phi_y L m, \end{aligned} \quad (10)$$

where $m = \cos\theta$, $n = \sin\theta$ represent the orientation of the typical fiber segment located between $\mathbf{X}_0$ and $\mathbf{X}_1$ and L(θ,x,y) represents the length of the fiber segment of orientation θ at position x,y in the network.

CONNECTION METHOD II

Method II employs the same functions f_1, g_1, h_1 to determine the microscopic quantities u_1', v_1' and ϕ_{10}'. An entirely different procedure, however, is employed to determine f_0, g_0 and h_0. A network connection element consisting of a typical bond, assumed to be located at $\mathbf{X}_0$, and the fiber segments emanating from the bond is considered. The displacements and rotations of the fiber segment ends away from the bond are prescribed by the functions f_1, g_1, h_1, f_2, g_2, h_2, etc. The displacements u_0, v_0 and ϕ_{0i} for each fiber segment may then be calculated in terms of the functions f_1, g_1, h_1, etc.; hence in terms of $u,v,u_x,u_y,v_x,v_y,\phi,\ \phi_x,\phi_y$. The computational effort required in this method is considerably greater than that of the preceding method and from a practical viewpoint is feasible only if the connection network element is relatively simple. For example, the connection element that consists of two straight fiber segments attached at their midpoints may prove to be a suitable one. If, in fact, the network were to be comprised of a repetitive array of fiber segments, this method would be highly acceptable as shown by Klemm and Wózniak (1970).

CONNECTION METHOD III

Method III incorporates additional degrees of freedom in the mathematical model, which permits less restrictive assumptions to be made concerning the relationship between the microscopic and macroscopic levels of deformation. It is assumed that the displacement of a fiber segment end at $\mathbf{X}_1$ can be expressed in the form

$$\begin{aligned} u_1' &= u_0(x,y) + \psi_1(x,y)x_1' \\ v_1' &= v_0(x,y) + \psi_2(x,y)y_1', \end{aligned} \quad (11)$$

where x′,y′ represent the location of a point at the microscopic level with the micro-

[2] Subscripts x, y for functions u, v, ϕ denote partial derivatives. Eq., $u_x \equiv \partial u/\partial x$, etc.

scopic level origin selected at a macroscopic point x,y. In a similar fashion the rotation ϕ_{10}' of the $\mathbf{X}_1$ end of the fiber segment situated between $\mathbf{X}_0$ and $\mathbf{X}_1$ could be expressed as

$$\phi_{10} = \phi_0(\theta, x, y) + \Phi_1(\theta, x, y)x_1' + \Phi_2(\theta, x, y)y_1' \, . \tag{12}$$

In addition to the previously defined field functions ϕ, u, v, we have now introduced the new field functions ψ_1, ψ_2, Φ_1, Φ_2. One specialized interpretation of these functions would be to select

$$\begin{aligned}
\psi_1 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x'} \, , \\
\psi_2 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y'} \, , \\
\Phi_1 &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x'} \, , \\
\Phi_2 &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial y'} \, .
\end{aligned} \tag{13}$$

Generally, the gradients $\partial x/\partial x'$ etc. would be assumed to be functions of position x,y. When the special case $\partial x/\partial x' = 1$, $\partial y/\partial y' = 1$, $\partial x/\partial y' = 0$, $\partial y/\partial x' = 0$ prevails, the theory that results from method III would reduce to that obtained from method I directly.

Field equations for a network in elastic equilibrium-connection method I

The field equations for determining the macroscopic response of the continuum are obtained from the principle of virtual work. That is, the change in elastic strain energy stored in the network is equal to the virtual work of the forces that act on the boundary of the network for an arbitrary set of virtual displacements of the field quantities. The appropriate methodology and equations are provided for the case of connection method I. Connection methods II and III require a slightly altered procedure; however, the basic methodology is the same for all connection methods.

The strain energy $W^F_{\mathbf{X}_0\mathbf{X}}$ of the fiber segment situated between points $\mathbf{X}_0$, $\mathbf{X}$ is given by

$$\begin{aligned}
W^F_{\mathbf{X}_0\mathbf{X}} &= \tfrac{1}{2}K_{11}\epsilon^2_{\mathbf{X}_0\mathbf{X}} + \tfrac{1}{2}K_{22}\delta^2_{\mathbf{X}_0\mathbf{X}} + \\
&\tfrac{1}{2}K_{33}\phi^2 + \tfrac{1}{2}K_{44}(\phi + \phi_x Lm + \phi_y Lm)^2 + \\
&K_{23}\phi\delta_{\mathbf{X}_0\mathbf{X}} + K_{24}(\phi + \phi_x Lm + \phi_y Lm)\delta_{\mathbf{X}_0\mathbf{X}} + \\
&K_{34}\phi(\phi + \phi_x Lm + \phi_y Lm) - B_1\Delta T\epsilon_{\mathbf{X}_0\mathbf{X}} - \\
&B_2\Delta T\delta_{\mathbf{X}_0\mathbf{X}} - B_3\Delta T\phi - B_4\Delta T(\phi + \phi_x Lm + \\
&\phi_y Lm) \, ,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\epsilon_{\mathbf{X}_0\mathbf{X}} &= L(m^2 u_x + m^2 v_y + \\
&mn(u_y + v_x)) \, , \\
\delta_{\mathbf{X}_0\mathbf{X}} &= L(m^2 v_x - m^2 u_y + \\
&mn(v_y - u_x)) \, ,
\end{aligned} \tag{15}$$

and where in general the fiber segment length L and the coefficients K_{11}, $K_{12}, \ldots,$ are functions of orientation angle θ and position x,y.

The fiber segments and bonds that comprise the network are prescribed in terms of the fiber distribution function $D_f(\theta,x,y)$ and the bond distribution function $D_B(x,y)$. The quantity $D_f d\theta$ represents the number of fiber segments lying between θ and $\theta + d_\theta$ per bond. The quantity $D_B dx dy$ represents the number of bonds in the network between x, x+dx and y, y+dy.

The elastic energy stored in a bond located at point (x,y) is assumed to be given by the function

$$W^B_X = \int_0^{2\pi} D_f G d\theta \tag{16}$$

where G, the bond energy storage function associated with fiber segments of orientation θ, is a function of segment orientation and position in the network, and depends upon the bond deformation. In general, G may depend upon ϕ, various order derivatives of ϕ with respect to θ, and possibly may be conceived to depend upon weighted integrals of ϕ over the interval $(0,2\pi)$. In the following, it will be assumed that G depends on θ,x,y,ϕ and ϕ_θ.

The elastic energy stored in the fiber segments at a bond located at point x,y is given by

$$W^F_{\mathbf{X}} = \int_0^{2\pi} W^F_{\mathbf{X}_0\mathbf{X}} D_f d\theta \qquad (17)$$

The total strain energy U stored in the network can be expressed as

$$U = \int_A W_X dA \qquad (18)$$

where

$$W_{\mathbf{X}} = D_B \left[\frac{1}{2} W^F_{\mathbf{X}} + W^B_{\mathbf{X}}\right] \qquad (19)$$

represents the strain energy per unit network area. Alternately, the strain energy U may be expressed as

$$U = \int_A \int_0^{2\pi} F(\theta, x, y, \phi, \phi_\theta, \phi_x, \phi_y, U_x, U_y, V_x, V_y)\, d\theta\, dx\, dy, \qquad (20)$$

where

$$F = \frac{1}{2} D_B D_F \left[W^F_{\mathbf{X}_0\mathbf{X}} + K_B \phi_\theta^2\right] \qquad (21)$$

is a function that explicitly depends on the independent variables θ, x,y and the field displacement and rotation functions u,v,ϕ and on the first partial derivatives u_x, u_y, v_x, v_y, ϕ_x, ϕ_y, ϕ_θ of the field functions.

The principle of virtual work can be written as the relation

$$\delta \int_A \int_0^{2\pi} F\, d\theta dA = \delta \int_B \left[T_X u + T_y v + \int_0^{2\pi} m^* \phi d\theta\right] ds, \qquad (22)$$

where the symbol δ represents the process of determining the first variation of the expression following it. The area A of the network is limited by a boundary B along which the forces in the x-direction T_x and in the y-direction T_y per unit boundary length are prescribed functions. Also, along the boundary B, the moment per unit angle θ per unit of boundary length denoted m^* is a prescribed function.

The variational condition must be true for arbitrary variations $\delta\phi$, δu, δv in the displacement and rotation functions, which must require that

$$\frac{\partial F_{u_x}}{\partial x} + \frac{\partial F_{u_y}}{\partial y} = 0,$$

$$\frac{\partial F_{v_x}}{\partial x} + \frac{\partial F_{v_y}}{\partial y} = 0, \qquad (23)$$

$$F_\phi - \frac{\partial F_{\phi_\theta}}{\partial \theta} - \frac{\partial F_{\phi_x}}{\partial x} - \frac{\partial F_{\phi_y}}{\partial y} = 0$$

at all points x, y in the plane of the network and that

$$\left[\int_0^{2\pi} F_{u_x} d\theta\right] \nu_x + \left[\int_0^{2\pi} F_{u_y} d\theta\right] \nu_y = T_x,$$

$$\left[\int_0^{2\pi} F_{v_x} d\theta\right] \nu_x + \left[\int_0^{2\pi} F_{v_y} d\theta\right] \nu_y = T_y, \qquad (24)$$

$$F_{\phi_x}\nu_x + F_{\phi_y}\nu_y = m$$

on boundary B of the network. The quantities $\boldsymbol{v}_x$, $\boldsymbol{v}_y$ represent the direction cosines of the unit vector $\boldsymbol{v}$, which is an outwardly directed vector normal to the boundary. In view of the form of the boundary conditions [24], one may define the "stress" components t_{xx}, t_{xy}, t_{yx}, t_{yy} as

$$t_{xx} = \int_0^{2\pi} F_{u_x} d\theta,$$

$$t_{yx} = \int_0^{2\pi} F_{u_y} d\theta, \qquad (25)$$

$$t_{xy} = \int_0^{2\pi} F_{v_x} d\theta,$$

$$t_{yy} = \int_0^{2\pi} F_{v_y} d\theta.$$

The quantities $F_{\phi_x} \equiv m_{xz}$ and $F_{\phi_y} \equiv m_{yz}$ represent physical couples per unit length as can be observed from the last boundary condition.

DISCUSSION

In the general case when connection method I or II is employed, the displacement functions u, v are functions of position x, y, and the rotation function ϕ is a function of position x, y and orientation θ. The equilibrium equations [23] along with the boundary conditions [24] or suitable conditions involving u, v, and ϕ on the boundary must be solved to determine the field functions u, v, and ϕ. In general, the problem is of the isoperimetric type since the strain energy must be made to assume an extreme value, while the conditions of specified stress on the boundary result in constraining relations of the rotation function ϕ of integral type. When the boundary conditions involve only the displacements u, v and the function ϕ, the variational problem is no longer isoperimetric.

The relations [25] are expressions between the stress components t_{xx}, t_{xy}, t_{yx}, t_{yy} and the displacement gradients u_x, u_y, v_x, v_y at a point. These relations are analogous to the stress-strain relations of conventional elasticity theory. In elasticity theory, one possible solution to the field equations results when the stresses and strains are homogeneous, i.e., when the stresses and strains are constant throughout the body. This type of situation may be used then to determine experimentally the elastic coefficients. It is not necessarily true that homogeneous solutions exist for arbitrary specification of the distribution functions L, D_B, and D_F. Nonetheless, one would presume that there are some distribution functions for which the homogeneous case prevails. Furthermore, this situation is of considerable interest in achieving an experimental verification of the theory.

One homogeneous solution case of interest is that of the "rigid" bond. Then the rotation function ϕ is a function of position x, y but is independent of orientation θ. The "stress-strain" equations may then be obtained from equations [25], and it is presumed that equations [23] are satisfied because all of the quantities F_{u_x}, F_{u_y} etc. are independent of position. It can be shown that the stress-strain relations so obtained are of the general form of those proposed by Askar and Cakmak (1968). As noted by these authors, the theory is of the same format as Eringen's (1966) linear theory of micropolar elasticity. If the network were assumed to be a repetitive array of straight beam elements, the theory employing connection method I would be analogous with Askar and Cakmak's (1968). However, if connection method II were employed and the network were a repetitive array, the resulting theory would be analogous to that of Klemm and Wózniak (1970), which again is of the general form of the micropolar elasticity.

A case of considerable interest with regard to the study of the stress-strain behavior of paper networks is the assumed homogeneous solution that permits variation of rotation function ϕ with orientation θ. In this situation the displacement gradients and the rotation function are independent of position. Thus, the equilibrium equations [23] reduce to

$$F_\phi - \frac{dF_{\phi_\theta}}{d\theta} = 0 \qquad (26)$$

along with the first two of the boundary conditions [24]. The boundary conditions can be used to express the displacement gradients in terms of an integral expression for the rotation function ϕ, which can be used in [26] to obtain a single equation for the rotation function ϕ. The resulting equation is in general an integro-differential equation with variable coefficients. This class of problems is analogous to Sternstein's (1971) isoperimetric variational problems of fiber networks.

When the bond energy function G is assumed to have a very special form corresponding to having each fiber segment pin connected at its end to a rigid member, but elastically restrained from free rotation with respect to the rigid member, the equilibrium equation [26] reduces to an algebraic form. In this case, the rotation function ϕ can be eliminated from the equations if the rotation of the rigid member is used instead. The resulting theory will then again be of the general form of Askar and Cakmak's (1968); however, now the bonds can be considered to have a degree of flexibility of their own.

The behavior of the paper network can be synthesized as a distribution of layers of repetitive arrays of fiber segments. This method may be referred to as the laminate method since the composite behavior may be thought of as a set of individual 2-dimensional fiber networks bonded together to form a sort of laminate. This approach is appealing from a computational point of view since the solution of equation [26] for a repetitive array degenerates to an algebraic equation when each layer is presumed to be characterized by a homogeneous deformation. From a physical point of view also, this method has considerable merit since it permits the sheet to be made up of laminates whose behavior varies with position through the thickness of the sheet. Connection method II is the logical choice for this method.

The use of connection method III in the form of [13] necessitates the determination of the unknown functions $\partial x/\partial x'$, $\partial x/\partial y'$, $\partial y/\partial x'$, $\partial y/\partial y'$ in addition to the field functions u, v, and ϕ. The theory that results may be compared with that of Klemm and Wózniak (1970), who introduced arbitrary parameters in their strain energy function to account for "distortion" of the plane of the network. Maye (1970) also theorized the necessity for incorporating functions of this type in order to provide for reasonable freedom between the microscopic and macroscopic deformation levels. Maye further postulated that the quantities $\partial x/\partial x'$, etc. should be constant unless the elastic properties of the medium are assumed not be homogeneous. It may further be suggested that the constant values of $\partial x/\partial x'$, etc. should be selected in such a way as to ensure that the macroscopic elastic symmetry has the desired form. Thus, if for the paper network one selects the distribution functions L, D_B, D_F on the basis of experimental observation of paper sheets, then the quantities $\partial x/\partial x'$, etc. could be partly determined from the condition that the sheet exhibit orthotropic elastic symmetry.

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