SOLUTION OF AN ORTHOTROPIC BEAM PROBLEM BY FOURIER SERIES[1]

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ABSTRACT

An exact solution of the characteristic fourth-order partial differential equation for plane orthotropic elasticity is obtained by using the Fourier method. Particular emphasis is given to the orthotropic beam subjected to a concentrated load. Stress distributions calculated from the resulting series are compared with those given by elementary bending theory.

INTRODUCTION

The first application of the Fourier method to the solution of isotropic beam problems was given by Ribiere in 1889. Further progress in this area was contributed by Filon (see Timoshenko and Goodier 1951). In particular, Filon specialized in the calculation of stress fields in isotropic materials subjected to symmetric concentrated loads. Similar problems have been discussed also by Timpe (Timoshenko and Goodier 1951). A study of the bending stresses in an isotropic beam using Fourier series was made by Goodier (1932) and a detailed discussion of the solutions is given by Timoshenko and Goodier (1951). Pickett (1944) has obtained a solution for the rectangular isotropic plate under two types of boundary loads: (1) loads that vary parabolically, and (2) concentrated loads. The general solution in Fourier series of the bending of an orthotropic beam under an arbitrary load distributed symmetrically was given by Lekhnitskii (1947). In this paper, we also take up the application of the Fourier method to the orthotropic beam problem—particularly, the simply supported beam under a concentrated load acting at arbitrary distance from the end of the beam. The solution is exact, and the stresses are expressed as a sine series convergent in the limit. The coefficients of the terms in the series are determined by using assigned boundary conditions. Mathematical analyses of stress distributions are illustrated. In particular, the variation of normal stress σ_{11}, over the cross section at various positions in a simply supported beam, long or short, is compared with that given by the elementary theory of bending.

BASIC EQUATIONS AND SOLUTION

The characteristic fourth-order partial differential equation for the orthotropic plane stress problem has the form (Jayne and Hunt 1969):

$$K_1\frac{\partial^4\phi}{\partial x_1^4} + K_2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2} + K_3\frac{\partial^4\phi}{\partial x_2^4} = 0, \quad (1)$$

where K_i is the compliances of the beam defined as

$$K_1 = S_{2222},\ K_2 = 2(S_{1122} + 2S_{1212}),\ \text{and}\ K_3 = S_{1111}.$$

One of the general solutions of equation (1) can be expressed in the form of a Fourier series as

$$\phi(x_1, x_2) = \sum_{n=1}^{\infty} [A_n \cosh\gamma_n\alpha x_2 + B_n \sinh\gamma_n\alpha x_2 + C_n \cosh\gamma_n\beta x_2 + D_n \sinh\gamma_n\beta x_2] \times \sin\gamma_n x_1, \quad (2)$$

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where $\gamma_n = n\pi/l$, and l is the length of the beam,

$$\left.\begin{aligned}\alpha &= \sqrt{\frac{\frac{K_2}{K_3} + \sqrt{\left(\frac{K_2}{K_3}\right)^2 - \frac{4K_1}{K_3}}}{2}},\\ \beta &= \sqrt{\frac{\frac{K_2}{K_3} - \sqrt{\left(\frac{K_2}{K_3}\right)^2 - \frac{4K_1}{K_3}}}{2}},\end{aligned}\right\} \text{real or imag.}$$

and A_n, B_n, C_n, D_n are arbitrary constants to be determined from boundary conditions. It follows from equation (2) that the stress components are

$$\begin{aligned}\sigma_{11} &= \frac{\partial^2\phi}{\partial x_2^2}\\ &= \sum_{n=1}^{\infty} \gamma_n^2 [\alpha^2(A_n \cosh \gamma_n \alpha x_2 + B_n \sinh \gamma_n \alpha x_2)\\ &\quad + \beta^2 (C_n \cosh \gamma_n \beta x_2\\ &\quad + D_n \sinh \gamma_n \beta x_2)] \sin \gamma_n x_1\end{aligned}$$

$$\begin{aligned}\sigma_{22} &= \frac{\partial^2\phi}{\partial x_1^2}\\ &= -\sum_{n=1}^{\infty} \gamma_n^2 [A_n \cosh \gamma_n \alpha x_2 + B_n \sinh \gamma_n \alpha x_2\\ &\quad + C_n \cosh \gamma_n \beta x_2\\ &\quad + D_n \sinh \gamma_n \beta x_2] \sin \gamma_n x_1\\ \sigma_{12} &= \frac{-\partial^2\phi}{\partial x_1 \partial x_2}\\ &= -\sum_{n=1}^{\infty} \gamma_n^2 [\alpha(A_n \sinh \gamma_n \alpha x_2\\ &\quad + B_n \cosh \gamma_n \alpha x_2)\\ &\quad + \beta (C_n \sinh \gamma_n \beta x_2\\ &\quad + D_n \cosh \gamma_n \beta x_2)] \times\\ &\quad \cos \gamma_n x_1. \qquad (3)\end{aligned}$$

Consider a simply supported orthotropic beam subjected to a concentrated load at an arbitrary point along the upper edge as shown in Fig. 1. The stress boundary conditions for this type of loading are given by

$\sigma_{22} = 0$	for $x_2 = h$	$0 \leqslant x_1 \leqslant (ml - a)$ and $(ml + a) \leqslant x_1 < l$	(a)
$\sigma_{22} = -P$ $= -2aq$	$x_2 = h$	$(ml - a) \leqslant x_1 \leqslant (ml + a)$	(b)
$\sigma_{22} = 0$	$x_2 = -h$	$0 \leqslant x_1 \leqslant l$	(c)
$\sigma_{12} = 0$	$x_1 = \pm h$	$0 \leqslant x_1 \leqslant l$	(d)
$\sigma_{11} = 0$	$x_1 = 0, l$	$\lvert x_2 \rvert \leqslant h$	(e)
$\int_{-h}^{h} \sigma_{12}\, dx_2$ $= (1 - m)P$	$x_1 = 0$		(f)
$= mP$	$x_1 = l$		(g) (4)

In these equations P is the total load and q is the load intensity. These two parameters are related by the expression $2aq = P$ as $a \to 0$; the parameter m is a fraction less than unity. Initially, we attempt to find an expression for stress σ_{22} that satisfies the boundary condition (4b). It is expressed in the form of a sine series as

$$\sigma_{22} = \sum_{n=1}^{\infty} a_n \sin \gamma_n x_1. \qquad (5)$$

The unknown coefficients a_n are determined by the usual methods of obtaining the coefficients of a Fourier series. Using the boundary conditions given by (4a) and (4b), we obtain

$$\sigma_{22} = -\frac{2P}{l} \sum_{n=1}^{\infty} \sin nm\pi \sin \gamma_n x_1, \qquad (6)$$

where the equality $P = 2aq$ has been used. Since shear stress, σ_{12}, must be zero at the edge $x_2 = \pm h$, for all values of x_1 it follows that

$$\begin{aligned}-\sum_{n=1}^{\infty} \gamma_n^2 [&\alpha(A_n \sinh \gamma_n \alpha h + B_n \cosh \gamma_n \alpha h)\\ &+ \beta (C_n \sinh \gamma_n \beta h + D_n \cosh \gamma_n \beta h)]\\ &\times \cos \gamma_n x_1 = 0, \qquad (7)\end{aligned}$$

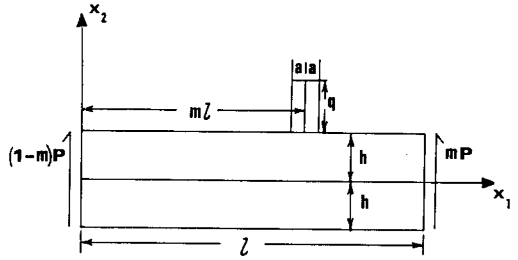

FIG. 1. Simply supported orthotropic beam subjected to a concentrated load.

$$-\sum_{n=1}^{\infty} \gamma_n^2 [\alpha(-A_n \sinh \gamma_n \alpha h + B_n \cosh \gamma_n \alpha h) + \beta(-C_n \sinh \gamma_n \beta h + D_n \cosh \gamma_n \beta h)] \times \cos \gamma_n x_1 = 0. \quad (8)$$

Substituting σ_{22} at $x_2 = -h$ into the second of (3) gives

$$-\sum_{n=1}^{\infty} \gamma_n^2 [A_n \cosh \gamma_n \alpha h - B_n \sinh \gamma_n \alpha h + C_n \cosh \gamma_n \beta h - D_n \sinh \gamma_n \beta h] \times \sin \gamma_n x_1 = 0. \quad (9)$$

Finally, substituting $\sigma_{22} = -(2P/l) \times \sum_{n=1}^{\infty} \sin n m \pi \sin \gamma_n x_1$ at $x_2 = h$ into the second of (3) gives

$$-\sum_{n=1}^{\infty} \gamma_n^2 [A_n \cosh \gamma_n \alpha h + B_n \sinh \gamma_n \alpha h + C_n \cosh \gamma_n \beta h + D_n \sinh \gamma_n \beta h] \times \sin \gamma_n x_1 = -\frac{2P}{l} \sum_{n=1}^{\infty} \sin n m \pi \sin \gamma_n x_1. \quad (10)$$

In general, $\sin \gamma_n x_1$ and $\cos \gamma_n x_1$ are nonzero. Therefore, if equations 7–10 are solved simultaneously, the unknown constants A_n, B_n, C_n, and D_n for each term of the Fourier series can be determined. They are as follows:

$$A_n = \frac{-P\beta \sinh \gamma_n \beta \sin n m \pi}{\gamma_n^2 l(\alpha \sinh \gamma_n \alpha h \cosh \gamma_n \beta h - \beta \cosh \gamma_n \alpha h \sinh \gamma_n \beta h)},$$

$$B_n = \frac{-P\beta \cosh \gamma_n \beta h \sin n m \pi}{\gamma_n^2 l(\alpha \cosh \gamma_n \alpha h \sinh \gamma_n \beta h - \beta \sinh \gamma_n \alpha h \cosh \gamma_n \beta h)},$$

$$C_n = \frac{P\alpha \sinh \gamma_n \alpha h \sin n m \pi}{\gamma_n^2 l(\alpha \sinh \gamma_n \alpha h \cosh \gamma_n \beta h - \beta \cosh \gamma_n \alpha h \sinh \gamma_n \beta h)},$$

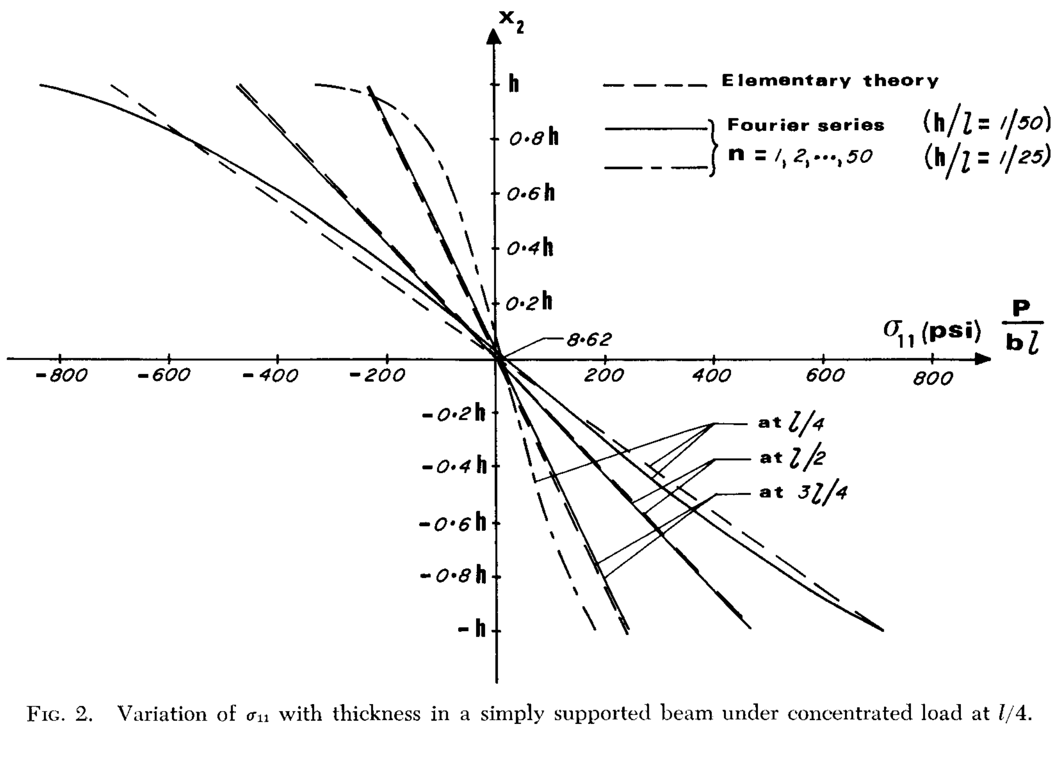

FIG. 2. Variation of σ_{11} with thickness in a simply supported beam under concentrated load at $l/4$.

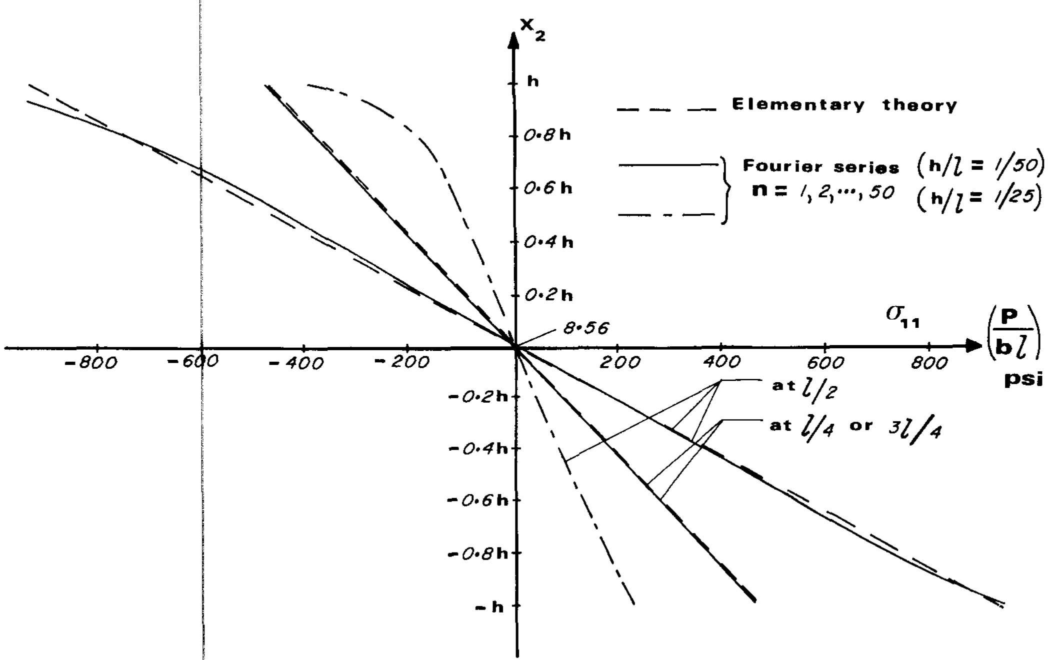

FIG. 3. Variation of σ_{11} with thickness in a simply supported beam under concentrated load at $l/2$.

$$D_n = \frac{P\alpha \cosh \gamma_n \alpha h \sin n m \pi}{\gamma_n^2 l(\alpha \cosh \gamma_n \alpha h \sinh \gamma_n \beta h - \beta \sinh \gamma_n \alpha h \cosh \gamma_n \beta h)}. \quad (11)$$

Thus, the problem is solved.

The boundary conditions for σ_{11} given by (4e) specify that σ_{11} be zero on the end surfaces of the beam. The first of equations 3 indicates that σ_{11} is a function of sin $\gamma_n x_1$. This function will be zero at $x_1 = 0, l$, and hence boundary condition (4e) is satisfied. The remaining boundary conditions given by (4f) and (4g) require that the reactions of the beam be carried in shear on the end surfaces. The integrals given by (4f) and (4g) are satisfied after substitution of σ_{12} as given by the third of (3). The integration is straightforward but tedious.

For the calculation of all stress, fifty terms of the series were used. Additional terms did not substantially modify the magnitude of the stress shown. The variation of normal stress σ_{11} over the cross section at various positions of a long or short beam is compared with the values given by elementary bending theory in Figs. 2, 3. A moderate departure from the linear distribution as given by elementary bending theory can be noted.

The variation of shear stress σ_{12} with both x_1 and x_2 is shown in Figs. 4, 5. The shear stress distribution through the cross section of the beam is modified substantially from that predicted by elementary theory. Furthermore, the variation of shear stress with x_1 at constant x_2 is not linear as is predicted by elementary theory. Finally, Fig. 6 shows the variation of σ_{22} along the x_1 axis of the beam. The reader is reminded that the results of σ_{22} and the effect of the length to the σ_{11} are unavailable from elementary theory. The result of the variation of σ_{22} along the x_1 axis is in general agreement with the result of an isotropic beam as given by Timoshenko and Goodier (1951).

CONCLUSIONS

For bending of an orthotropic beam under an arbitrarily distributed load, one can sometimes use the polynomial stress function. (Jayne and Tang 1970). If the load is distributed in a more complicated manner, however, especially in cases where the load

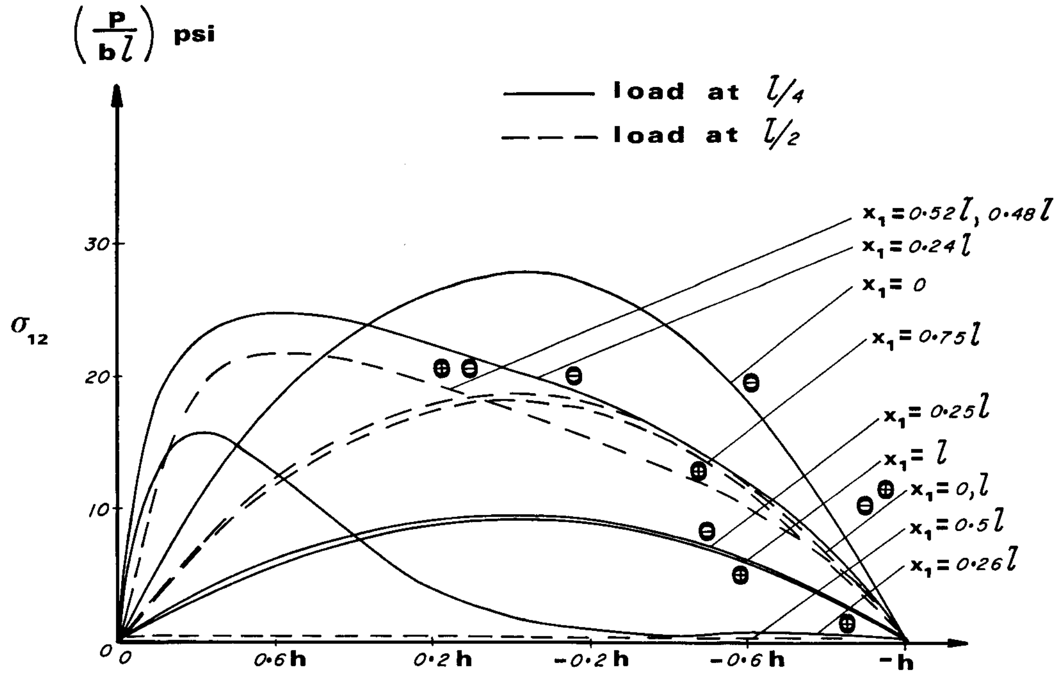

FIG. 4. Distribution of σ_{12} through the cross section of a beam ($h/l = 1/50$) under concentrated load.

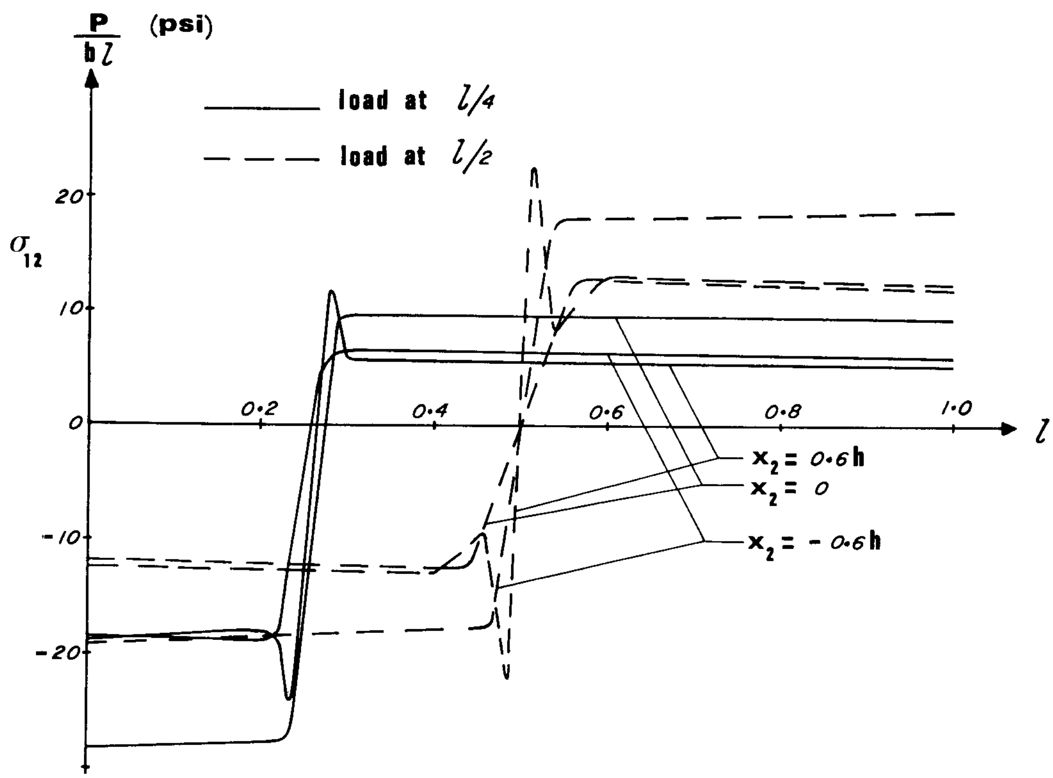

FIG. 5. Variation of σ_{12} along x_1 axis of a beam ($h/l = 1/50$) under concentrated load.

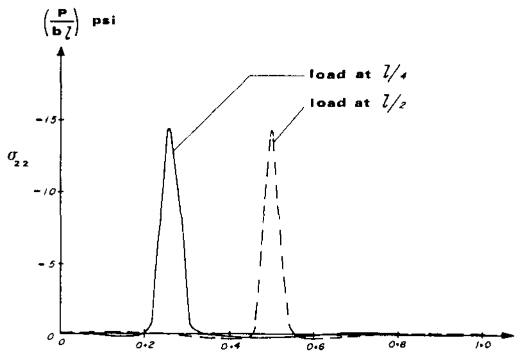

FIG. 6. Variation of σ_{22} along x_1 axis at $x_2 = 0$ $(l/2 = 1/50)$.

extends over only part of the beam, such as the concentrated forces considered in this paper, the stress distribution can be obtained readily with Fourier series. This method will provide a solution that satisfies all boundary conditions.

Elementary bending theory is based on the assumption of a linear distribution of normal stress σ_{11} at every section. It has been shown in Figs. 2, 3 that the elementary Bernoulli-Euler theory of bending is very accurate if the thickness of the beam is small in comparison with its length. The variation of normal stress σ_{11} over some cross sections of an orthotropic beam calculated from the first fifty terms of the series is nonlinear. However the departure from linearity is moderate and for most applications can probably be ignored. On the other hand, the distribution of shear stress, σ_{12}, through the cross section of the beam differs markedly from that predicted by elementary theory. It may be necessary to account for this difference in some type of design. The normal stress component, σ_{22}, which is not accessible by means of elementary theory, exhibits an unusual variation along the length of the beam. Although of theoretical interest for most applications, the variation of σ_{22} can be ignored.

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