PLANE STRESS AND PLANE STRAIN IN ORTHOTROPIC AND ANISOTROPIC MEDIA[1]

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ABSTRACT

Using the concepts embodied in the equations of stress equilibrium, strain compatibility and Hooke's law, the partial differential equations of plane stress and plane strain characteristic of homogeneous orthotropic bodies were derived. The plane stress problem requires the simultaneous solutions of five differential equations. Normally only one of the equations, that requiring compatibility of strain in the plane, is solved. In contrast, the plane strain problem requires the solution of but one differential equation.

INTRODUCTION

In this paper, the plane stress and plane strain problems of orthotropic and anisotropic elasticity are derived in detail with particular emphasis given to organization. Basically, this subject has been well developed and reviewed by Lekhnitskii (1963) and Hearmon (1961). However, because of the general unfamiliarity of today's wood scientists with this topic, the authors feel that a consolidated, detailed, and readable treatment is needed. Furthermore, it is intended that this article will furnish a ready reference or starting point for subsequent papers on solution of specific problems involving wood and woodbase materials that meet the conditions defined herein.

Since but a handful of three-dimensional elastic problems in isotropic as well as anisotropic media are amenable to analytic solution, it is common to specialize problems to the simpler two-dimension approximations. In many practical problems, the magnitude and variation of stresses and strains along one of the coordinate axes are small and can be ignored with minimal error. Consequently, in many instances the two-dimensional simplifications of three-dimensional problems provide sufficient information for design purposes.

The plane stress and plane strain problems of elasticity are classic examples of idealized two-dimensional problems. A body, extended in two directions (the plane) and of small dimension in the third direction, frequently can be considered to be in a state of plane stress. For this problem, boundary loads are restricted entirely to the plane of the body. In contrast, a body which is extended along one axis and which is subject to boundary loads oriented in a direction normal to the axis can sometimes be characterized as in a state of plane strain. The boundary loads must be independent of the long axis. Methods of attacking problems of this type are well documented in a number of elementary as well as advanced texts on elastic theory (Shames 1964; Sokolnikoff 1956; Timoshenko 1951; Wang 1953).

The formulation of both problems culminates in a set of partial differential equations which must be solved subject to prescribed conditions at the boundary of the body. In general, a function which will satisfy all the differential equations of either problem is difficult, if not impossible, to find. However, by ignoring all but one of the differential equations in each case, approximate solutions can be obtained.

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$$\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{2311} & 2S_{2322} & 2S_{2333} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1311} & 2S_{1322} & 2S_{1333} & 4S_{1323} & 4S_{1313} & 4S_{1312} \\ 2S_{1211} & 2S_{1222} & 2S_{1233} & 4S_{1223} & 4S_{1213} & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} \tag{2}$$

THE FUNDAMENTAL EQUATIONS OF ELASTIC THEORY

Four fundamental sets of equations are necessary for the solution of a problem in the mechanics of a continuous body. These are: 1) a law relating stress and strain, 2) the stress equilibrium equations, 3) the strain compatibility equations, and 4) stress boundary equations. In the formulation of the plane stress and plane strain problems, the first three sets of equations, reduced to a simple form, are combined to yield the partial differential equations to be solved. The fourth set of equations are used to establish the boundary conditions for particular problems.

In this paper, the orthotropic and anisotropic bodies are considered to be linearly elastic. Consequently, Hooke's law is used as the basic equation relating stress and strain. This law is conveniently expressed in matrix form as either of equations (1) or (2).

In these equations, the S_{ijkl} are coefficients of the compliance tensor, whereas the C_{ijkl} are the coefficients of the stiffness tensor. Rather than writing a tensor equation for Hooke's law, we have employed its matrix equivalent in these equations. Using matrix symbolism, we can write the equations

$$[\gamma] = [S][\sigma]$$

and

$$[\sigma] = [C][\gamma]$$

The **S** and **C** matrices are related by the equation

$$[S] = [C]^{-1} \tag{3}$$

Thus, if the coefficients of either the **C** or **S** matrix are known, the coefficients of the other can be calculated by matrix inversion.

For an orthotropic material, Hooke's law assumes a much simpler form. Strains are expressed as functions of the stresses by the matrix equation (4), whereas the inverse relationship is given by equation (5). For purposes of calculation, it is sometimes convenient to write equation (4) in terms

$$\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{2211} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{3311} & S_{3322} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & \frac{-\nu_{13}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{21}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{31}}{E_1} & \frac{-\nu_{32}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{6}$$

of the engineering elastic coefficients, ν_{ij}, E_i, and G_{ij}. With these constants, equation (4) takes the form of equation (6). In equation (6), the indices 1, 2, and 3 refer to the orthotropic axes of the material. For example, E_1 is Young's modulus in the x_1 direction. A similar interpretation holds for moduli E_2 and E_3. The coefficient ν_{ij} (Poisson's coefficient) relates normal strains in orthogonal directions by the equation

$$\gamma_{ii} = -\nu_{ij}\gamma_{jj}$$

Specifically

$$\gamma_{11} = -\nu_{12}\gamma_{22}$$

The application of a strain γ_{22} induces a strain γ_{11} as given by the equation above. The coefficients G_{ij} are the shear moduli in the three orthotropic planes.

The elastic coefficient matrices of equations (5) and (6) are, or of course, related by matrix inversion. Inversion of the compliance matrix of equation (6) provides the coefficients C_{ijkl} of equation 5. These coefficients are as follows in equation (7).

It is to be emphasized that Hooke's law of equations (4) through (6) is applicable only to orthotropic boundary-value problems in which the geometric and orthotropic axes of the material are coincident. If, for

$$
\begin{aligned}
C_{1111} &= \frac{E_1(1-\nu_{32}\nu_{23})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}} \\
C_{1122} = C_{2211} &= \frac{E_1(\nu_{12}+\nu_{32}\nu_{13})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}} \\
C_{1133} = C_{3311} &= \frac{E_1(\nu_{13}+\nu_{12}\nu_{23})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}} \\
C_{2222} &= \frac{E_2(1-\nu_{13}\nu_{31})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}} \\
C_{2233} = C_{3322} &= \frac{E_2(\nu_{23}+\nu_{13}\nu_{21})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}} \\
C_{3333} &= \frac{E\,(1-\nu_{12}\nu_{21})}{1-2\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{31}-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}}
\end{aligned}
\tag{7}
$$

example, a straight-grained wood beam with the plane of the growth rings parallel and perpendicular to the faces of the board is under study, the beam can be considered as approximately orthotropic. If, on the other hand, a slope of grain exists in the board or the growth rings are not normal to one pair of surfaces, a higher degree of anisotropy is introduced. In situations where this occurs, one can use the law for transforming a fourth-order Cartesian tensor to obtain the elastic coefficients of the board in its geometric frame of reference. The transformation for components of the compliance tensor takes the form

$$\bar{S}_{ijkl} = a_{im}a_{jn}a_{ko}a_{lp}S_{mnop} \tag{8}$$

In this equation, the S_{mnop} are the compliances for the orthotropic axes, whereas the $\bar{S}_{ijkl}$ are the compliances for the geometric axes of the member. The a_{ij} are the direction cosines which relate the two sets of coordinates. Transformations of components of the stiffness tensor are effected by a similar law. Transformations of this type for orthotropic materials have been reviewed by Jayne and Suddarth (1966). As a result of the non-coincidence of orthotropic and geometric axes, it may be necessary to represent the elastic nature of an orthotropic body by Hooke's law in the form of equation (1). On the other hand, if the orthotropic and geometric axes are coincident, Hooke's law in the form of equation (4) or its engineering equivalent, equation (6), is used.

The second set of fundamental equations required for formulation of plane stress and plane strain problems are designated as the stress equilibrium equations. These equations can be written conveniently as a matrix expression which takes the form

$$
\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}
\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}
+
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = 0 \tag{9}
$$

In the absence of body forces (B_1, B_2, and B_3), equation (9) takes the expanded form

$$
\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} &= 0 \\
\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} &= 0 \\
\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} &= 0
\end{aligned}
\tag{10}
$$

Equilibrium in the form of equation (10) will be used exclusively in this paper.

Strain compatibility equations, which are the third set of requisite equations, can be represented conveniently by a set of six second-order partial differential equations which relate the six independent components of strain. These equations are based on the fundamental assumption that the components of displacement and their higher order derivatives are continuous functions of the coordinates. Relationships between the strain components result. Strain compatibility is given by the set of equations

$$
\begin{aligned}
\frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^2 \gamma_{11}}{\partial x_2^2} + \frac{\partial^2 \gamma_{22}}{\partial x_1^2} \\
\frac{\partial^2 \gamma_{13}}{\partial x_1 \partial x_3} &= \frac{\partial^2 \gamma_{11}}{\partial x_3^2} + \frac{\partial^2 \gamma_{33}}{\partial x_1^2} \\
\frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} &= \frac{\partial^2 \gamma_{22}}{\partial x_3^2} + \frac{\partial^2 \gamma_{33}}{\partial x_2^2} \\
2\frac{\partial^2 \gamma_{11}}{\partial x_2 \partial x_3} &= \frac{\partial}{\partial x_1}\left[\frac{\partial \gamma_{12}}{\partial x_3} + \frac{\partial \gamma_{13}}{\partial x_2} - \frac{\partial \gamma_{23}}{\partial x_1}\right] \\
2\frac{\partial^2 \gamma_{22}}{\partial x_1 \partial x_3} &= \frac{\partial}{\partial x_2}\left[\frac{\partial \gamma_{12}}{\partial x_3} + \frac{\partial \gamma_{23}}{\partial x_1} - \frac{\partial \gamma_{13}}{\partial x_2}\right] \\
2\frac{\partial^2 \gamma_{33}}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_3}\left[\frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{23}}{\partial x_1} - \frac{\partial \gamma_{12}}{\partial x_3}\right]
\end{aligned}
\tag{11}
$$

The final set of equations required for solution of elastic problems are commonly tagged as the stress boundary conditions. The equations require the components of stress to assume certain values at the boundary in accordance with the loading conditions on the body. The equations are conveniently written in matrix form as

$$
\begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}
\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{12}
$$

In this equation, the σ_{ij} are the components of stress, the n_i are the direction cosines of the outward normal of a bounding surface, and the T_i are the vector components of the force intensities applied at the boundary. As indicated earlier, this set of equations is used to establish the boundary conditions for the characteristic partial differential equations of plane stress and plane strain.

THE PLANE STRESS PROBLEM

For the plane stress problem, we consider a system of the type illustrated in Fig. 1. The body is of relatively small dimension along the X_3 coordinate and is extended in the 1–2 plane. It is of thickness h and the plane formed by the X_1 and X_2 axes lies midway between the upper and lower surfaces. As a first approximation, it can be considered a plane body referenced to the 1–2 plane. Forces applied at the boundary of the body also are located in the 1–2 plane.

Since the plane surfaces of the body are not subject to loading, it follows that the normal stress σ_{33} and shear stresses σ_{31} and σ_{32} must disappear at the surfaces $x_3 = \pm h/2$. It is assumed that these components of stress within the body are small in magnitude and as a first approximation can be considered to vanish at all interior points. As a result of symmetry of the stress tensor, i.e. $\sigma_{31} = \sigma_{13}$ and $\sigma_{23} = \sigma_{32}$, only three independent non-zero components of stress, σ_{11}, σ_{22}, and σ_{21}, remain. Since the body is loaded only in the plane, it is logical to assume that these stress components are functions only of x_1 and x_2. Consequently, the equilibrium equations (10) take on a particularly simple form

$$
\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} &= 0 \\
\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} &= 0
\end{aligned}
\tag{13}
$$

In order to develop the characteristic partial differential equations of plane stress and plain strain, we define a function $\phi(x_1, x_2)$ which is continuous and possesses pure and mixed derivatives which are continuous to at least the fourth order. The function ϕ is defined in such a manner as to satisfy the equilibrium equations as given by (13).

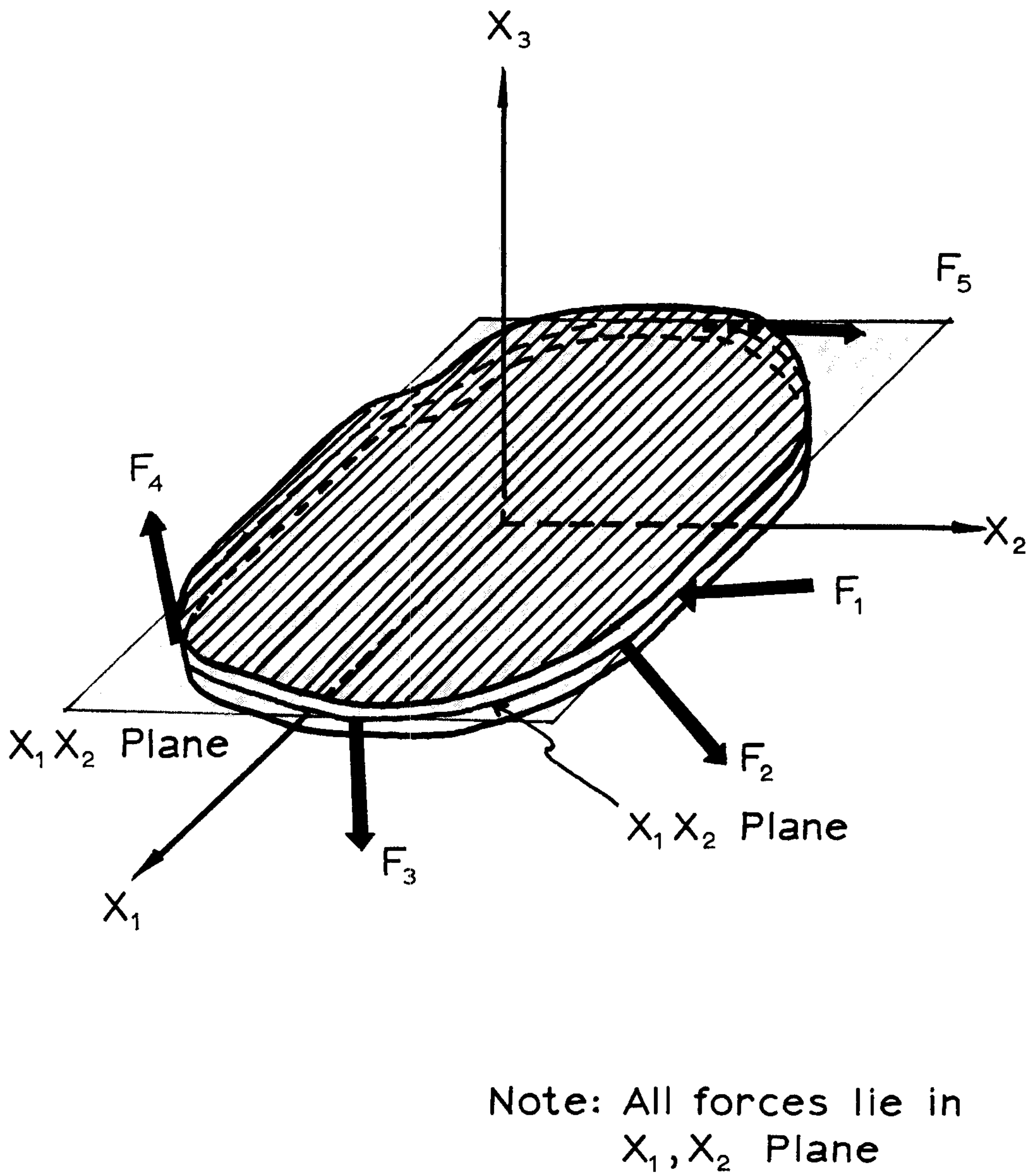

FIG. 1. A body subject to a state of plane stress.

As a result, the components of stress are given by

$$\begin{aligned} \sigma_{11} &= \frac{\partial^2 \Phi}{\partial x_2^2} \\ \sigma_{22} &= \frac{\partial^2 \Phi}{\partial x_1^2} \\ \sigma_{12} &= \frac{-\partial^2 \Phi}{\partial x_1 \partial x_2} \end{aligned} \tag{14}$$

By substituting equations (14) and (13), we can demonstrate readily that $\phi(x_1, x_2)$ as defined will satisfy the simplified equilibrium equations.

After substitution of the non-vanishing components of stress, σ_{11}, σ_{22}, and σ_{12}, into Hooke's law for the anisotropic material (equation [1]), it is apparent that all six independent components of strain exist in the anisotropic plane body. In contrast, if the three independent planar stresses are substituted into Hooke's law for the orthotropic solid (either of equations [4] or [6]), it is found that only four components of the strain tensor exist: γ_{11}, γ_{22}, γ_{33}, and γ_{12}. In either case (anisotropic or orthotropic), it is clear that plane stress is not accompanied by a state of plane strain.

Consider the anisotropic plane system. In the presence of only three components of stress, equations (1) simplify to

$$
\begin{aligned}
\gamma_{11} &= S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + 2S_{1112}\sigma_{12} \\
\gamma_{22} &= S_{2211}\sigma_{11} + S_{2222}\sigma_{22} + 2S_{2212}\sigma_{12} \\
\gamma_{33} &= S_{3311}\sigma_{11} + S_{3322}\sigma_{22} + 2S_{3312}\sigma_{12} \\
\gamma_{23} &= 2S_{2311}\sigma_{11} + 2S_{2322}\sigma_{22} + 4S_{2312}\sigma_{12} \\
\gamma_{13} &= 2S_{1311}\sigma_{11} + 2S_{1322}\sigma_{22} + 4S_{1312}\sigma_{12} \\
\gamma_{12} &= 2S_{1211}\sigma_{11} + 2S_{1222}\sigma_{22} + 4S_{1212}\sigma_{12}
\end{aligned} \qquad (15)
$$

The strains as given by this set of equations are functions only of x_1 and x_2 since the stresses on which the strains depend are functions only of x_1 and x_2. Consequently, the strain compatibility equations (11) reduce to a more simplified form. The simplified set, written in the same order as (11), is as follows:

$$
\begin{aligned}
\frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^2 \gamma_{11}}{\partial x_2^2} + \frac{\partial^2 \gamma_{22}}{\partial x_1^2} \\
0 &= \frac{\partial^2 \gamma_{33}}{\partial x_1^2} \\
0 &= \frac{\partial^2 \gamma_{33}}{\partial x_2^2} \\
\frac{\partial}{\partial x_1}\left[\frac{\partial \gamma_{13}}{\partial x_2} - \frac{\partial \gamma_{23}}{\partial x_1}\right] &= 0 \\
\frac{\partial}{\partial x_2}\left[\frac{\partial \gamma_{23}}{\partial x_1} - \frac{\partial \gamma_{13}}{\partial x_2}\right] &= 0 \\
\frac{\partial^2 \gamma_{33}}{\partial x_1 \partial x_2} &= 0
\end{aligned} \qquad (16)
$$

This set of strain compatibility equations results from the fact that none of the components of strain are functions of x_3.

The plane stress problem for an anisotropic body is formulated by combining equations (14), (15), and (16). The process of combination is straightforward and can readily be accomplished by substituting (14) into (15) followed by substitution of the new equations into equations (16). A set of six fourth-order homogeneous partial differential equations result, as given by equation (17).

The solution of this set of differential equations, even for the most simple boundary conditions, proves to be an extremely difficult task. As a result, it is conventional to consider that the first of this set of equations specifies completely the plane stress problem for the anisotropic body. The remaining five equations are not considered in the description of the problem. The error introduced by ignoring five of the equations is considered to be minimal. Consequently, the plane stress problem in an anisotropic body is characterized approximately by the partial differential equation (18) which must be solved subject to prescribed boundary conditions. These conditions frequently take the form of boundary stresses which, according to equations (14), are specified by second-order derivatives of the function ϕ.

The orthotropic plane stress problem is formulated in a manner similar to that followed for the anisotropic case. Hooke's law for the orthotropic plate, subjected to a state of plane stress, takes a somewhat more simple form than that of the anisotropic material. For the orthotropic medium, we have

$$
\begin{aligned}
\gamma_{11} &= S_{1111}\sigma_{11} + S_{1122}\sigma_{22} \\
\gamma_{22} &= S_{2211}\sigma_{11} + S_{2222}\sigma_{22} \\
\gamma_{33} &= S_{3311}\sigma_{11} + S_{3322}\sigma_{22} \\
\gamma_{12} &= 4S_{1212}\sigma_{12}
\end{aligned} \qquad (19)
$$

Because of the dependence of the strains on x_1 and x_2 only and, the nonexistence of

$$
\begin{aligned}
&S_{2222}\frac{\partial^4\Phi}{\partial x_1^4} - 4S_{1222}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} + 2(S_{1122} + 2S_{1212})\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} - 4S_{1112}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} + S_{1111}\frac{\partial^4\Phi}{\partial x_2^4} = 0 \\
&2S_{3322}\frac{\partial^4\Phi}{\partial x_1^4} - 2S_{3312}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} + 2S_{3311}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} = 0 \\
&2S_{3322}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} - 2S_{3312}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + 2S_{3311}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} = 0 \\
&2S_{3311}\frac{\partial^4\Phi}{\partial x_2^4} - 2S_{3312}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} + S_{3322}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} = 0 \\
&-2S_{2322}\frac{\partial\Phi}{\partial x_1^4} + 2(S_{1322} + 2S_{2312})\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} - 2(S_{2311} + 2S_{1312})\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + 2S_{1311}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} = 0 \\
&\text{and} \\
&-2S_{1311}\frac{\partial^4\Phi}{\partial x_2^4} + 2(S_{2311} + 2S_{1312})\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} - 4S_{2312}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + 2S_{2322}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} - 2S_{1322}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} = 0
\end{aligned}
\tag{17}
$$

$$
S_{2222}\frac{\partial^4\Phi}{\partial x_1^4} - 4S_{1222}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} + 2(S_{1122} + 2S_{1212})\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} - 4S_{1112}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} + S_{1111}\frac{\partial^4\Phi}{\partial x_2^4} = 0 \tag{18}
$$

shear strains γ_{13} and γ_{23}, two of the strain compatibility equations vanish identically. The four remaining compatibility equations are as follows:

$$
\begin{aligned}
\frac{\partial^2\gamma_{12}}{\partial x_1\partial x_2} &= \frac{\partial^2\gamma_{11}}{\partial x_2^2} + \frac{\partial^2\gamma_{22}}{\partial x_1^2} \\
\frac{\partial^2\gamma_{33}}{\partial x_1^2} &= 0 \\
\frac{\partial^2\gamma_{33}}{\partial x_1\partial x_2} &= 0 \\
\frac{\partial^2\gamma_{33}}{\partial x_2^2} &= 0
\end{aligned}
\tag{20}
$$

Combining equations (14), (19), and (20) in a manner similar to that followed for the plane anisotropic problem yields a set of fourth-order partial differential equations characteristic of the plane orthotropic problem.

These equations are as in equation (21).

A complete solution of the plane orthotropic problem must encompass these four equations. In a manner analogous to the anisotropic problem, it is conventional to disregard the last three equations of (20) and assume that the orthotropic problem is characterized by the first equation. Accordingly, we consider equation (22) as the characteristic differential equation of plane stress in an orthotropic medium.

THE PLANE STRAIN PROBLEM

For the plane strain problem, we consider a system of the type illustrated in Fig. 2. The body is of prismatic shape and is extended along the X_3 axis. Surface forces are functions of x_1 and x_2 only. These forces are restricted to act in a direction normal to the X_3 axis. We assume that the three components of body force, B_1, B_2, and B_3, are zero. The ends of the body are constrained. Consequently, the displacement component $u_3 = 0$ at the ends. Furthermore, because of symmetry of the body, $u_3 = 0$ at the midpoint $X_3 = 0$. As a result, it is assumed as a first approximation that $u_3 = 0$ everywhere. Since the boundary forces which act on the body are independent of x_3, it is assumed also that the displacement components u_1 and u_2 a large distance from the ends are functions of x_1 and x_2 only.

$$
\begin{aligned}
&S_{2222}\frac{\partial^4\Phi}{\partial x_1^4} + 2(S_{1122} + 2S_{1212})\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + S_{1111}\frac{\partial^4\Phi}{\partial x_2^4} = 0 \\
&S_{3311}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + S_{3322}\frac{\partial^4\Phi}{\partial x_1^4} = 0 \\
&S_{3311}\frac{\partial^4\Phi}{\partial x_1\partial x_2^3} + S_{3322}\frac{\partial^4\Phi}{\partial x_1^3\partial x_2} = 0 \\
&S_{3311}\frac{\partial^4\Phi}{\partial x_2^4} + S_{3322}\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} = 0
\end{aligned}
\tag{21}
$$

$$
S_{2222}\frac{\partial^4\Phi}{\partial x_1^4} + 2(S_{1122} + 2S_{1212})\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + S_{1111}\frac{\partial^4\Phi}{\partial x_2^4} = 0 \tag{22}
$$

As a result of the restrictions and approximations described above, it is apparent from an inspection of the equation for strains, expressed as partial derivatives of the displacements, that three components of strain are non-zero and three vanish. The three non-zero strains are γ_{11}, γ_{22}, and γ_{12}, whereas the three components of strain which vanish are γ_{33}, γ_{23}, and γ_{13}. Consequently, as a first approximation, the body is subject to a state of strain confined to the 1–2 plane.

For the case of plane strain, Hooke's law for the anisotropic material takes the matrix form:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1112} \\
C_{2211} & C_{2222} & C_{2212} \\
C_{3311} & C_{3322} & C_{3312} \\
C_{2311} & C_{2322} & C_{2312} \\
C_{1311} & C_{1322} & C_{1312} \\
C_{1211} & C_{1222} & C_{1212}
\end{bmatrix}
\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{12} \end{bmatrix}
$$

Since the strains γ_{11}, γ_{22}, and γ_{12} are functions only of x_1 and x_2, it follows that all six components of stress are similarly functions of x_1 and x_2. Consequently, in the absence of body forces, the stress equilibrium equations (9) take the simplified form

$$
\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} &= 0 \\
\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} &= 0 \\
\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} &= 0
\end{aligned}
\tag{23}
$$

Inspection of the strain compatibility equations (11) indicates that all are identically zero, with the exception of

$$
\frac{\partial^2\gamma_{11}}{\partial x_2^2} + \frac{\partial^2\gamma_{22}}{\partial x_1^2} = \frac{\partial^2\gamma_{12}}{\partial x_1\partial x_2}
$$

Substituting the strains into this equation using the compliance form of Hooke's law for the anisotropic solid yields the equation

$$
\begin{aligned}
&2S_{1211}\frac{\partial^2\sigma_{11}}{\partial x_1\partial x_2} + 2S_{1222}\frac{\partial^2\sigma_{22}}{\partial x_1\partial x_2} + 2S_{1233}\frac{\partial^2\sigma_{33}}{\partial x_1\partial x_2} + 4S_{1223}\frac{\partial^2\sigma_{23}}{\partial x_1\partial x_2} + 4S_{1213}\frac{\partial^2\sigma_{13}}{\partial x_1\partial x_2} \\
&+ 4S_{1212}\frac{\partial^2\sigma_{12}}{\partial x_1\partial x_2} = S_{1111}\frac{\partial^2\sigma_{11}}{\partial x_2^2} + S_{1122}\frac{\partial^2\sigma_{22}}{\partial x_2^2} + S_{1133}\frac{\partial^2\sigma_{33}}{\partial x_2^2} + 2S_{1123}\frac{\partial^2\sigma_{23}}{\partial x_2^2} \\
&+ 2S_{1113}\frac{\partial^2\sigma_{13}}{\partial x_2^2} + 2S_{1112}\frac{\partial^2\sigma_{12}}{\partial x_2^2} + S_{2211}\frac{\partial^2\sigma_{11}}{\partial x_1^2} + S_{2222}\frac{\partial^2\sigma_{22}}{\partial x_1^2} + S_{2233}\frac{\partial^2\sigma_{33}}{\partial x_1^2} \\
&+ 2S_{2223}\frac{\partial^2\sigma_{33}}{\partial x_1^2} + 2S_{2213}\frac{\partial^2\sigma_{13}}{\partial x_1^2} + 2S_{2212}\frac{\partial^2\sigma_{12}}{\partial x_1^2}
\end{aligned}
$$

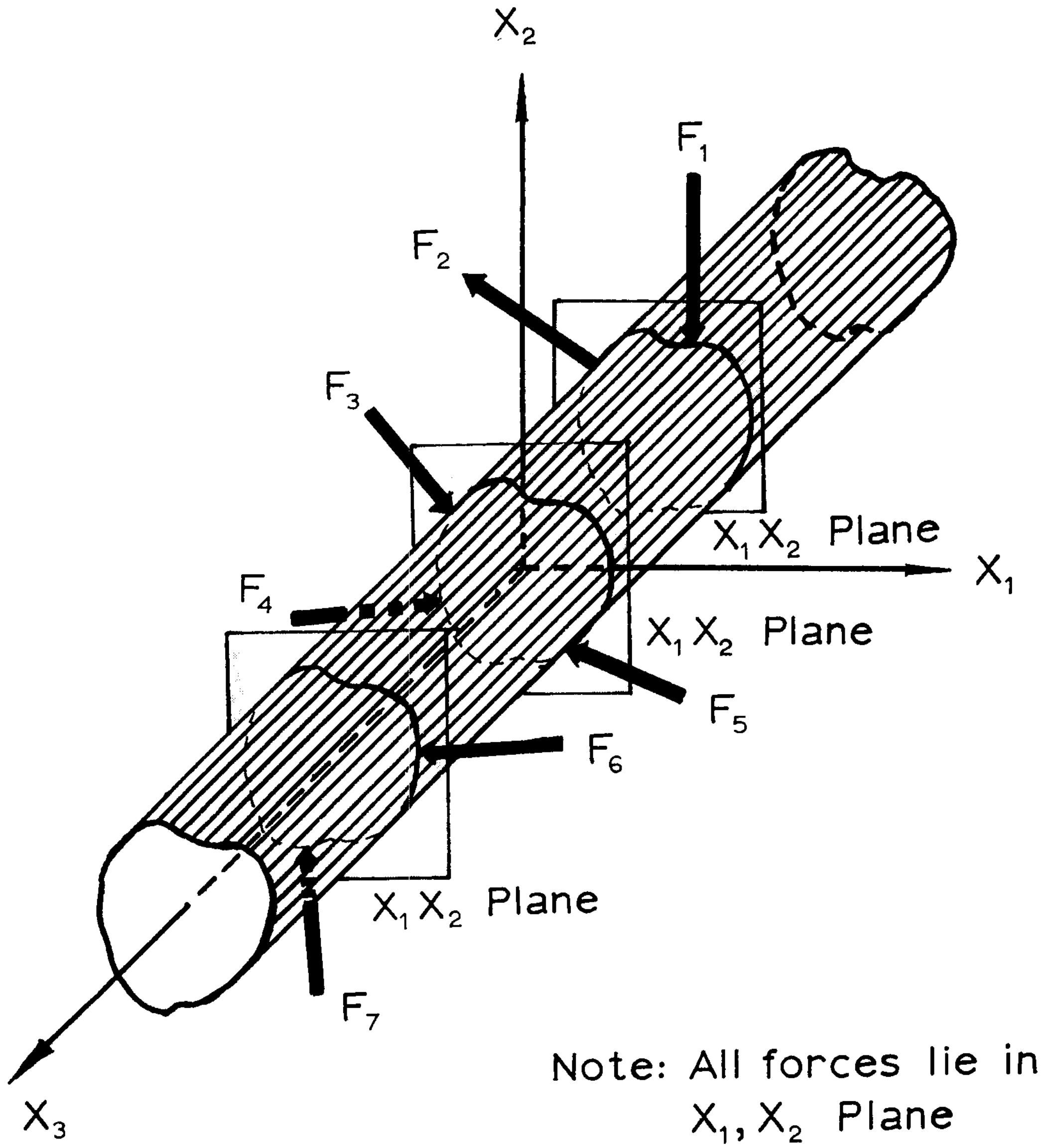

FIG. 2. A body subject to a state of plane strain.

The aim of this analysis, as was the case for plane stress, is to obtain a partial differential equation in a single dependent variable which characterizes plane strain in the anisotropic body. This requires the generation of an unknown function which satisfies the equilibrium equations (23). A simple function such as that defined by equation (14) will not suffice. A combination of unknown functions is required. As yet, this is unavailable, and the plane strain problem of anisotropic elasticity cannot be characterized by a homogeneous partial differential equation.

In contrast, the state of plane strain in the orthotropic medium can be described by means of a single partial differential equation. Substituting the state of plane strain into equation (4) for the orthotropic system yields the matrix equation

$$\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ 0 \\ 0 \\ 0 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{2211} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{3311} & S_{3322} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

which, when expanded, gives

$$\gamma_{11} = S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33}$$

$$\gamma_{22} = S_{2211}\sigma_{11} + S_{2222}\sigma_{22} + S_{2233}\sigma_{33}$$

$$\gamma_{33} = S_{3311}\sigma_{11} + S_{3322}\sigma_{22} + S_{3333}\sigma_{33} = 0$$

$$\gamma_{23} = 4S_{2323}\sigma_{23} = 0$$

$$\gamma_{13} = 4S_{1313}\sigma_{13} = 0$$

$$\gamma_{12} = 4S_{1212}\sigma_{12}$$

The fourth and fifth of this set of equations imply that

$$\sigma_{13} = \sigma_{23} = 0$$

whereas, the third equation can be solved for σ_{33} to yield

$$\sigma_{33} = \frac{-(S_{3311}\sigma_{11} + S_{3322}\sigma_{22})}{S_{3333}}$$

Substitution of this expression for σ_{33} into the first two of the equations for Hooke's law gives equation (24). These two equations with the simple expression for

$$\gamma_{12} = 4S_{1212}\sigma_{12}$$

indicate that the stresses σ_{11}, σ_{22}, σ_{12}, and σ_{33} are functions only of x_1 and x_2. Furthermore, as shown above, σ_{13} and σ_{23} are zero for the orthotropic medium.

In the absence of body forces, the stress equilibrium equations (9) take on a particularly simple form

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} = 0$$
$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = 0 \tag{25}$$

As was the case for plane stress, a function such as that described by equation (14) will satisfy these equations.

As was shown earlier, five of the six strain compatibility equations are identically zero for a state of plane strain. Substituting equations (24) and (25) into the one remaining strain compatibility equation

$$\frac{\partial^2\gamma_{11}}{\partial x_2} + \frac{\partial^2\gamma_{22}}{\partial x_1} = \frac{\partial^2\gamma_{12}}{\partial x_1 \partial x_2}$$

and using the stress function defined by equations (14), we arrive at the partial differential equation (26). It should be noted

$$\gamma_{11} = S_{1111} - \left(\frac{S_{1133}^{\ 2}}{S_{3333}}\right)\sigma_{11} + S_{1122} - \left(\frac{S_{1133}S_{3322}}{S_{3333}}\right)\sigma_{22}$$
$$\gamma_{22} = S_{2211} - \left(\frac{S_{2233}S_{3311}}{S_{3333}}\right)\sigma_{11} + S_{2222} - \left(\frac{S_{2233}^{\ 2}}{S_{3333}}\right)\sigma_{22} \tag{24}$$

$$\left(S_{2222} - \frac{S_{2233}^{2}}{S_{3333}}\right)\frac{\partial^4\Phi}{\partial x_1^4} + \left(2S_{1212} + 2S_{1122} - \frac{S_{1133}S_{3322}}{S_{3333}}\right)\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + \left(S_{1111} - \frac{S_{1133}^{2}}{S_{3333}}\right)\frac{\partial^4\Phi}{\partial x_2^4} = 0 \qquad (26)$$

that equation (26) which describes the plane strain problem in orthotropic materials has basically the same form as equation (22), which described plane stress in the orthotropic material. The difference between the two equations is that due only to the numerical constants.

CONCLUSIONS

The state of plane stress in both anisotropic and orthotropic media is described by sets of fourth-order homogeneous partial differential equations. Approximation to the stress distribution in either type of body can be obtained by considering only the strain compatibility equation relating the two normal strains and the shear strain in the plane. The homogeneous partial differential equation which results requires solution subject to the boundary conditions of the body.

Plane strain in the anisotropic body cannot be expressed in the form of a differential equation in a single unknown variable. In contrast, all five of the strain compatibility equations are identically zero for plane strain in an orthotropic body. Substituting a stress function into the remaining strain compatibility equation provides a fourth-order homogeneous equation. Consequently, plane strain in the orthotropic body is subject to exact solution.

REFERENCES

HEARMON, R. F. S. 1961. Applied anisotropic elasticity. Oxford Univ. Press. London.

JAYNE, B. A. AND S. K. SUDDARTH. 1966. Matrix-tensor mathematics in orthotropic elasticity. Orientation effects in the mechanical behavior of anisotropic structural materials. A.S.T.M. Spec. Tech. Pub. 405. Philadelphia, Pennsylvania.

LEKHNITSKII, S. G. 1963. Theory of elasticity of an anisotropic body. Holden-Day, Inc., San Francisco.

SHAMES, I. S. 1964. Mechanics of deformable solids. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

SOKOLNIKOFF, I. S. 1956. Mathematical theory of elasticity. McGraw-Hill, New York.

TIMOSHENKO, S. P. 1953. Theory of elasticity. McGraw-Hill, New York.

WANG, C. T. 1953. Applied elasticity. McGraw-Hill, New York.

FUKADA, E. 1968. Piezoelectricity as a fundamental property of wood. *Wood Sci. & Tech.* 2(4): 299–307. The piezoelectric effect in wood was related to the uniaxial orientation of cellulose crystallites in fibers and their monoclinic symmetry. A shear stress in one plane along the grain produced electrical polarization perpendicular to it. The value of the piezoelectric modulus, approximately 0.05 of that of a quartz crystal, was increased by chemical treatments that transform cellulose I to II or III. However, gamma irradiation high enough to decrease the molecular weight had little influence on the piezoelectric modulus. Variation of the phase angle between sinusoidal stress and polarization with temperature showed a maximum of advanced phase near room temperature and a maximum of delayed phase at about –100 C. The former was caused by dielectric loss of water at temperatures above freezing and the latter by the viscoelastic loss from local vibrations of cellulose molecules. (A)