# A MATHEMATICAL ANALYSIS TO DETERMINE THE VOLUME OF RESIDUES AND LUMBER PRODUCED IN THE SAWMILL EDGING PROCESS

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(Received June 1989)

ABSTRACT

Mathematical models were developed to determine volumes of the different types of materials produced in edging hardwood lumber. Six edging patterns that consider the complete or partial removal of wane were analyzed, and mathematical models were separately developed for each pattern. The models were incorporated into a computerized sawmill simulation program, and several logs were computer sawn and processed using a wide range of edging patterns. Results from these initial runs appeared reasonable and realistic when compared to empirical sawmill data. The edging models developed in this study may be useful to analysts studying sawmill fiber balances as well as lumber recovery and grade trade-offs. In addition, incorporation of the edging models into sawmill simulation and analytical programs can represent a significant and important step in the evolution of computerized sawmill analysis.

*Keywords:* Hardwoods, sawmill, simulation, edging, modeling, edging products.

## INTRODUCTION

Work on the development of a computer simulation of a hardwood sawmill led to a mathematical method by which volume determinations could be obtained for all of the materials produced at the headsaw, including sawdust, slabs, and flitches produced at each cut (Savsar and Kersavage 1982). This procedure also enabled the determination of the volume of the other sawmill materials such as edgings, trims, sawdust, and lumber produced at each pass through the sawmill's edgers and trimmers.

The simulation program that incorporated these mathematical methods for volume determination was limited initially to one method of sawing and edging.

*Wood and Fiber Science,* 22(3), 1990, pp. 271–282

Edging was restricted to a method that removed all of the wane. The mathematics to solve for the volumes produced using this edging method are not very complicated. However, a general solution to this problem, one in which any edging cut can be made whether in the wane or not, is considerably more difficult.

Development of a general solution for the mathematical analysis of the edging process has several important implications. A general solution to this problem could bring sawmill and edging analysis a step closer to actual situations, thereby leading to more reliable and accurate analyses. It could also widen the scope of problems that could be studied and solved using sawmill computer analysis, such as sawmill fiber balance and lumber recovery and grade trade-offs. In addition, a mathematical solution to the general edging problem could lay the foundation for the development of more complex edging models, such as those used to describe or simulate tapered and asymmetrical edging and various multiple-saw edging systems.

No general solution to the problem of determining volumes of materials produced in edging flitches cut from logs assumed to be in the shape of truncated cones could be found in the literature. However, some related work has been done in Europe (Ionaitis 1979; Koperkin 1971; Krutel 1969). Although this European research made some attempts at mathematically describing some sawmill products such as cants and cant faces, this research did not include any description of a generalized solution to the overall edging problem. A recent mathematical study on lumber yield from logs did not include any equations for determining volumes of edging materials produced in processing the logs (Zheng et al. 1989). The primary purpose of this paper is to describe the development of mathematical models, which can be used to determine volumes of materials produced under a variety of edging methods that might be employed in a hardwood sawmill.

## ASSUMPTIONS

In developing the mathematical models described in this paper, the following major assumptions were made:

1. The logs used in this study were assumed to be in the shape of perfect truncated cones.

2. All saw cuts, including slab and flitch cuts, in the log breakdown, were made parallel to the central or longitudinal axis of the log.

3. The faces on flitches sawn from logs described in assumption 1, were assumed to be trapezoidal in shape. While it is recognized that this assumption is a simplification of the theoretically more accurate flitch face of a truncated hyperbola, the trapezoidal flitch face assumption has the advantage of allowing a considerable reduction in the complexity of the edging analysis models, while still closely approximating the theoretically more accurate flitch face configuration (Savsar 1982).

Other assumptions pertaining to the development of the aforementioned edging models are stated elsewhere in this paper.

## DERIVATION OF EDGING MODELS

The development of a single overall mathematical model to determine volumes of materials produced in a sawmill under different edging practices poses some difficult problems. Not only do the flitches possess a complex shape, which is difficult to define mathematically, but the edgings produced from these flitches have an even more complex configuration, which varies in shape and dimension with edging pattern, flitch width, and kerf thickness. The variability in the shape and size of the edgings is compounded by the fact that flitch dimensions and wane configurations will vary as a function of flitch location within the log, flitch thickness, log taper, log diameter, and log length, thereby leading to even more edging variability. Still another complicating factor is the fact that most kerfs produced in the edging process occur in the wane, and they are also difficult to describe mathematically.

As a result of these complexities, no single mathematical model or general solution was developed that could be used to determine edging volumes produced under a variety of possible conditions. Instead the various possible edging patterns were analyzed and classified into several types of edging patterns, and different models were separately developed that could be used for each pattern.

An analysis of the various possible edging patterns obtainable, using a conventional two-saw edger, indicated there are six distinct edging patterns that require a separate volume model (Fig. 1). With these models, volumes for all other combinations of edging methods could be solved. The models that were developed assumed that the two edging cuts made to edge any given flitch were parallel to each other and to the central axis of the flitch. It was further assumed that edging cuts were symmetrically made on each flitch. The solution to asymmetrical edging is a related problem that is presently beyond the scope of this paper.

The model to determine volumes for edging shown in Pattern 1 assumes that the edging kerfs do not fall anywhere within the wane portion of the flitch. Volume of edgings ($V_{E_1}$) in this case can be obtained from the following relationship:

$$V_{E_1} = V_F - (V_{L_1} + V_{SD_1}) \tag{1}$$

where the terms $V_F$, $V_{L_1}$ and $V_{SD_1}$ are as defined in the list of notations shown in Table 1. Some of the terms described in this table are also diagrammatically illustrated in Figs. 2 and 3.

The volume of any given flitch ($V_F$) is given by the equation

$$V_F = \ell\Bigg[\frac{z_1}{2}(r^2 - z_1^2)^{1/2} + \frac{r^2}{2}\sin^{-1}\frac{z_1}{r} - \left(\frac{z_2}{2}(r^2 - z_2^2)^{1/2} + \frac{r^2}{2}\sin^{-1}\frac{z_2}{r}\right)$$
$$+ \frac{z_1}{2}(R^2 - z_1^2)^{1/2} + \frac{R^2}{2}\sin^{-1}\frac{z_1}{R} - \left(\frac{z_2}{2}(R^2 - z_2^2)^{1/2} + \frac{R^2}{2}\sin^{-1}\frac{z_2}{R}\right)\Bigg] \tag{2}$$

as developed earlier by Savsar and Kersavage (1982). The volumes of lumber and sawdust ($V_{L_1}$ and $V_{SD_1}$) produced in edging using Pattern 1 is defined by

$$V_{L_1} = t\ell(w'_i - 2K_E) \tag{3}$$

and

$$V_{SD_1} = 2(tK_E\ell). \tag{4}$$

For Pattern 2 it is assumed that the edging kerfs on the narrow or small end of the flitch fall within the wane portion of the flitch, whereas the edging kerfs on the wide or large end fall somewhere within the flitch's trapezoidal face. In this edging pattern it is important to note the facial configurations on the top and

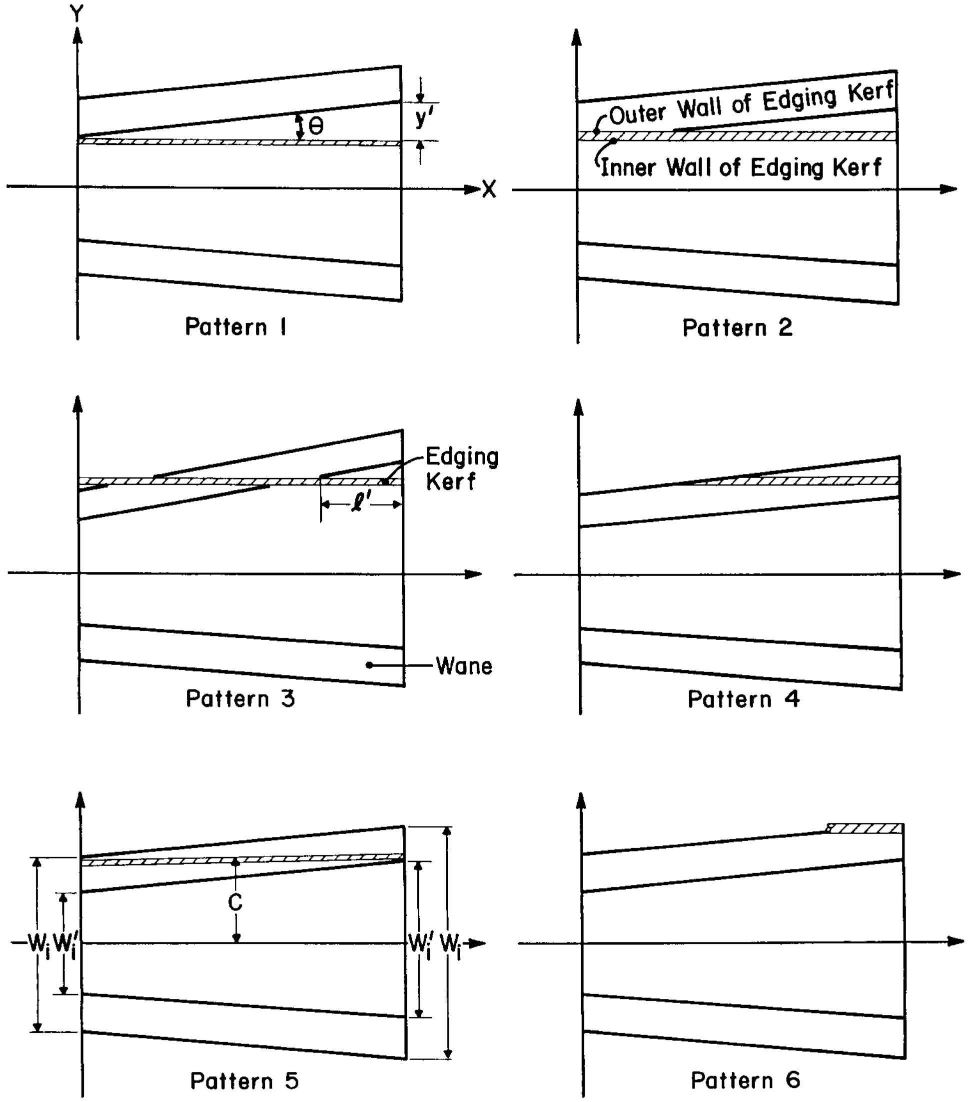


FIG. 1. Schematic flitch diagrams illustrating the various basic edging patterns considered in this study. All the patterns are illustrated using the small trapezoidal face. Although only one edging kerf is shown on the narrow or bark face in each of the six edging patterns illustrated, the edging kerf not shown would have the same relative location on the opposite side of each flitch. Pattern 1 illustrates an edging method in which all wane is removed but none is sawn. In Pattern 2 all wane is also removed but the edging kerf is located such that it emerges on the wane-wood interface on the narrow end of the flitch. In Pattern 3 the edging kerf on the wide end is located within the woody portion of the flitch face, and emerges in the wane and ends somewhere outside of it on the narrow end. Pattern 4 is similar except the kerf is located within the wane on the wide end. In Pattern 5 the entire kerf lies within the wane. In Pattern 6 the outer boundary line (outer wall) of the edging kerf on the wide end of the flitch intersects with the junction of the outer boundary line of the wane and the end line at the wide end of flitch. Notations are defined in Table 1 and elsewhere in text.

TABLE 1. *Notations for parameters used to develop mathematical models for determining the volumes of sawmill products.*

| Notation | Units | Parameter |
|---|---|---|
| $V_{E_j}$ | in.$^3$ | Volume of edgings produced using jth edging pattern |
| $V_F$ | in.$^3$ | Volume of flitch |
| $V_{L_j}$ | in.$^3$ | Volume of lumber produced using jth edging pattern |
| $V_{SD_j}$ | in.$^3$ | Volume of sawdust produced using the jth edging pattern |
| $V'_{L_j}$ | in.$^3$ | Volume of edged and trimmed lumber using jth edging pattern |
| $V_{SDT_j}$ | in.$^3$ | Volume of sawdust produced in trimming lumber edged using jth edging pattern |
| $V_{T_j}$ | in.$^3$ | Volume of trims produced in trimming lumber edged using jth edging pattern |
| t | in. | Flitch or lumber thickness |
| $\ell$ | in. | Untrimmed length of flitch or lumber |
| $\ell'$ | in. | Distance on the x-axis, on any given parallel plane on or within the flitch, between the outer kerf wall-wane intersection point and an outer kerf wall-end line intersection point on the large end of the flitch |
| W | in. | Maximum width of flitch face on any parallel plane within a flitch |
| w | in. | Minimum width of flitch face on any parallel plane within a flitch |
| $w'_i$ | in. | Minimum width on small or top trapezoidal face of the ith flitch |
| $w_i$ | in. | Minimum width on large or bottom trapezoidal face of the ith flitch |
| $W'_i$ | in. | Maximum width on small or top trapezoidal face of the ith flitch |
| $W_i$ | in. | Maximum width on large or bottom trapezoidal face of the ith flitch |
| $K_E$ | in. | Edger kerf thickness |
| y' | in. | End line distance on y-axis (large end), on any given parallel plane within the flitch, between the lines defined by the outer wall of the edging kerf and the wane line |
| c | in. | The distance on the y-axis between the outer wall of the edging kerf and the flitch center line on any given flitch plane |
| r | in. | Radius of debarked log, small end |
| R | in. | Radius of debarked log, large end |
| z | in. | Distance from origin on z-axis of any given sawn face as it was originally located within the log |
| $z_1$ | in. | Distance from origin on the z-axis of the bark-side face of any given flitch as it was originally located within a log |
| $z_2$ | in. | Distance from origin the z-axis of the pith-side face of any given flitch as it was originally located within a log |
| $z'_2$ | in. | Point on z-axis in a plane parallel to the surface where $\ell'$ is equal to $\ell$ and where the faces within an edging change from triangular shapes to trapezoidal |
| $\Theta$ | degs. | Angle of log taper |
| $A_1$ | in.$^2$ | Surface area of a trapezoidal-shaped edging face |
| $A_2$ | in.$^2$ | Surface area of a triangular-shaped edging face |

bottom surfaces of the edgings produced using this method. The top or smaller surface of the edging has triangular facial configurations, whereas the bottom or larger surface has a trapezoidal-shaped face. It is also important to note that in the population of planes parallel to the top and bottom surfaces, there is a transition from the triangular-shaped face to the trapezoidal face as you progress from the top surface to the bottom one. This transition occurs on a plane where $\ell'$ is equal to $\ell$ and is defined by the point $z'_2$ on the z-axis. This point's coordinates on the y-z plane are $(c, z'_2)$ where

$$c = \frac{W}{2} - y' \tag{5}$$

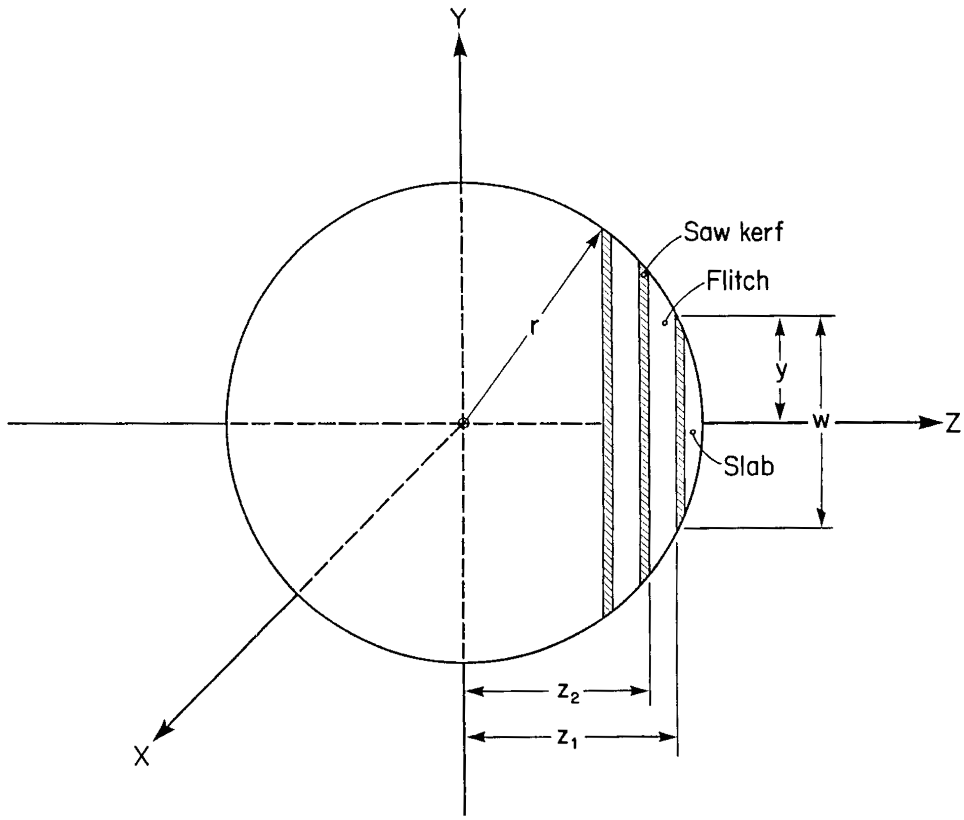


FIG. 2. View of small end of a sawlog illustrating notations and three-dimensional coordinates used in denoting geometry and relative locations of materials produced using a simulated kerf-centered sawing method. The origin is located at the small end of the log and the x-axis runs through the center of the log. Notations are defined in Table 1 and elsewhere in text.

and

$$z'_2 = \sqrt{r^2 - c^2} \tag{6}$$

In the development of the volume calculation model for Pattern 2, the following relationship should also be noted:

$$\frac{y'}{\ell'} = \left(\frac{\frac{W}{2} - \frac{w}{2}}{\ell}\right) \tag{7}$$

and therefore that

$$\ell' = \frac{2y'\ell}{W - w}. \tag{8}$$

It should also be noted that y' is a variable, since W is a variable in the relationship where

$$y' = \frac{W}{2} - c. \tag{9}$$

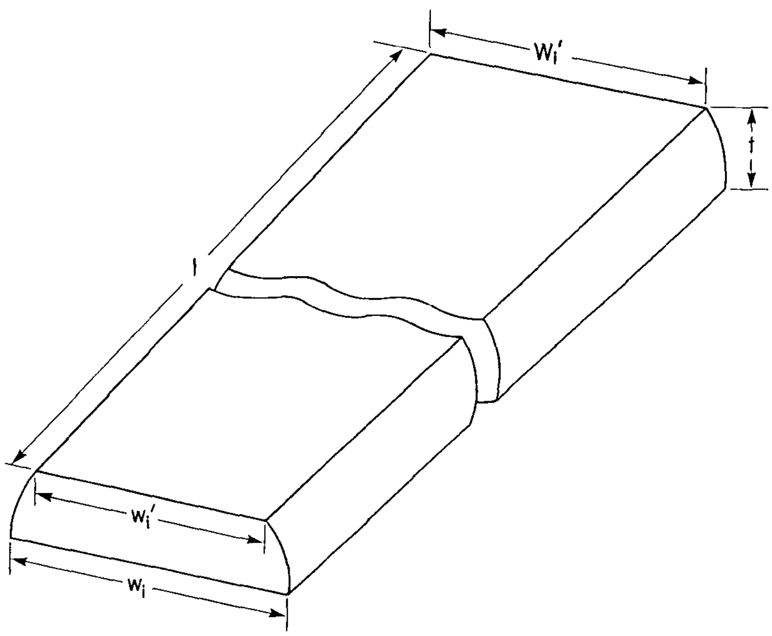


FIG. 3. Illustration of a typical unedged flitch produced from a log with a truncated cone shape. Notations are defined in Table 1.

In order to use these equations in solving for edging material volumes, it is necessary to know either $\ell'$ or c, since either one can be derived from the other. If $\ell'$ is given, then by using Eqs. (5) and (7), the following relationship for c can be obtained:

$$c = \frac{W}{2} - \ell'\left(\frac{\frac{W}{2} - \frac{w}{2}}{\ell}\right) \tag{10}$$

Conversely if c is known, then by using Eqs. (8) and (9), the following equation for $\ell'$ can be derived:

$$\ell' = \left(\frac{W}{2} - c\right)\left(\frac{2\ell}{W - w}\right) \tag{11}$$

In order to find the volume of a Pattern 2 edging, it is necessary to integrate the edging surface from $z_2$ to $z_1$. However, since the surface changes its shape at $z'_2$ from triangular to trapezoidal, it is necessary to perform two integration steps, one for the trapezoidal-shaped portion of the edging and the other for the triangular-shaped section. The volumes that are obtained by these integrations are then added together to obtain the volume of the edging.

In determining the volume ($V_1$) of the trapezoidal-shaped edging portion, it is first noted that the trapezoidal surface area ($A_1$) is given by

$$A_1 = \left(\frac{y_1 + y_2}{2}\right)\ell \tag{12}$$

where $y_1$ and $y_2$ are the end line distances (on the y-axis) of the outer wall of the edging kerf from the wane on the small and large ends of the flitch, respectively.

The following two relationships should also be noted:

$$y_1 = (w/2 - c) \tag{13}$$

and

$$y_2 = (W/2 - c). \tag{14}$$

The volume $V_1$ can then be found as follows:

$$V_1 = \int_{z_2}^{z'_2} A_1\, dz = \int_{z_2}^{z'_2} \frac{\left(\frac{w}{2} - c\right) + \left(\frac{W}{2} - c\right)}{2}\, \ell\, dz \tag{15}$$

and also

$$V_1 = \int_{z_2}^{z'_2} \frac{\sqrt{r^2 - z^2} + \sqrt{R^2 - z^2} - 2c}{2}\, \ell\, dz \tag{16}$$

and finally

$$V_1 = \frac{\ell}{2}\left(\frac{z}{2}\sqrt{r^2 - z^2} + \frac{r^2}{2}\sin^{-1}\frac{z}{r} + \frac{z}{2}\sqrt{R^2 - z^2} + \frac{R^2}{2}\sin^{-1}\frac{z}{R} - 2cz \Bigg|_{z_2}^{z_2'}\right). \tag{17}$$

In determining the volume ($V_2$) of the triangular-shaped section of the Pattern-2 edging, the triangular-shaped surface area ($A_2$) is defined by the following relationship:

$$A_2 = (y'/2)\ell' \tag{18}$$

and since

$$y' = (W/2) - c \tag{19}$$

and then under the condition that $\Theta$ is constant

$$\ell' = y'/\tan\Theta \tag{20}$$

it follows that

$$A_2 = y'^2/2 \tan\Theta. \tag{21}$$

The volume $V_2$ can then be determined using the following relationships:

$$V_2 = \int_{z'_2}^{z_1} A_2\, dz = \int_{z'_2}^{z_1} \frac{y'^2}{2\tan\Theta}\, dz \tag{22}$$

However, since

$$y'^2 = (W/2 - c)^2 = (W/2)^2 - 2c(W/2) + c^2 \tag{23}$$

and where, by definition as shown in Fig. 2,

$$W = 2\sqrt{R^2 - z^2} \tag{24}$$

then $V_2$ takes the form

$$V_2 = \int_{z'_2}^{z_1} \frac{(R^2 - z^2) - 2c\sqrt{R^2 - z^2} + c^2}{2 \tan \Theta} dz \tag{25}$$

and finally

$$V_2 = \frac{1}{2 \tan \Theta}\left[(R^2 + c^2)z - \frac{z^3}{3} - 2c\left(\frac{z}{2}\sqrt{R^2 - z^2} + \frac{R^2}{2}\sin^{-1}\frac{z}{R}\right)\Bigg|_{z'_2}^{z_1}\right]. \tag{26}$$

Assuming that two identical edgings will be produced from each flitch that is edged, the total volume $V_{E_2}$ of edging material produced using Pattern 2 would then be given by the following:

$$V_{E_2} = 2(V_1 + V_2). \tag{27}$$

Edging Pattern 3 results only in triangular-faced edgings, and therefore Eqs. (25) and (26) also apply here. However, the integration limits are not $z_2'$ to $z_1$ but are $z_2$ to $z_1$. The total volume of edgings produced using Pattern 3 would be equal to twice the volume obtained using Eq. (26).

The methods used to determine the volume of edgings produced using Pattern 4 is similar to those used for Pattern 3 except that the integration limits for Eqs. (25) and (26) would be changed from $z'_2$ to $z_1$ to $z_2$ to $z'_c$, where $z'_c$ is defined as follows:

$$z'_c = (R^2 - c^2)^{0.5}. \tag{28}$$

In Pattern 5, volume determination of edgings is similar to that in Pattern 2 with respect to trapezoidal faces. Equations (16) and (17) apply here also. However, integration limits have to be changed from $z_2$ to $z'_c$ in these equations.

In Pattern 6 there are no edgings since all the cut is converted into sawdust, and the volume of this sawdust is calculated using the same equations as were used in Pattern 4 and the sawdust volume methods described below.

The models developed to determine the volumes of edgings ($V_{E_j}$) produced from any of the six edging patterns shown in Fig. 1 can also be used to determine the exact volume of sawdust ($V_{SD_j}$) produced in any given edging cut. In determining sawdust volume, it is important to note that the volume of any given edging $\left(\frac{V_{E_j}}{2}\right)$ was obtained from material external to the outer wall of the edging kerf. If a new volume of edging $\left(\frac{V_{E_j}'}{2}\right)$ is determined using the same mathematical model but beginning from the internal wall of the edging kerf, edging sawdust volume can be obtained using the following relationship:

$$V_{SD_j} = 2\left(\frac{V'_{E_j}}{2} - \frac{V_{E_j}}{2}\right). \tag{29}$$

The volume of lumber ($V_{L_j}$) produced from edging any given flitch can be obtained using the following relationship:

$$V_{L_j} = V_F - (V_{SD_j} + V_{E_j}) \quad (30)$$

and finally the volume $V'_{L_j}$ of any given specimen of lumber after edging and trimming is given by the following:

$$V'_{L_j} = V_{L_j} - (V_{SDT_j} + V_{T_j}). \quad (31)$$

The edging models described in this paper can also be employed to determine trim volumes in sawmill component studies. In this approach each piece of trim would be considered to be a lumber specimen with a length equal to the trim length. Volumes of sawdust and trims produced in the trimming process would then be obtained using the same methods and equations that were used to determine the volume of edgings, edging sawdust, and edged lumber. A detailed description of the specific methods and models that could be used to calculate trim product volumes is beyond the scope of this paper.

## TESTING AND APPLICATION OF EDGING MODELS

The models developed and described in this paper to calculate volumes of edging materials for the various edging patterns were incorporated into a sawmill simulation program (Savsar and Kersavage 1982) and were used to computer saw a variety of different logs. Comparing the simulation results with those from several empirical studies, it was concluded that the sawmill simulation program, which employed the edging models developed in this paper, validly depicted actual sawmilling conditions.

To illustrate an example of one of the possible uses of the edging models, several computer runs using four different edging methods were generated. The results of these simulated log breakdown and edging operations are summarized in Table 2.

It should be noted that the four different edging patterns used in the simulated sawing runs are not necessarily those that would be commonly used in practice. They were selected primarily to demonstrate the effect that a wide range of edging practices, ranging from severe (i.e., all wane removed) in Pattern 1 to conservative (none of the flitch face removed), might have on the yields of sawmill products.

Analysis of the results of these initial trials indicates that the edging component data generated by the use of the edging models did not appear to be unusual. Results from this computerized edging experiment indicated that actual board-foot yield of lumber increased from 103 to 133 as the edging pattern became more conservative regarding removal of wane. This represented an increase in lumber recovery of 29%. Conversely, the amount of chippable material dropped from 20% to 6%.

## SUMMARY AND CONCLUSION

In summary, mathematical models were developed to determine volumes of materials produced in edging under a variety of different conditions. Six different edging patterns were identified and defined, which encompassed all of the possible methods of complete or partial removal of wane in the edging process. Mathematical models were then developed that could be used to solve for the volumes of edging products produced from use of each one of the six edging patterns. Use of these models would enable the determination of the volumes of all possible

TABLE 2. *Simulated sawmill product recovery data using four different edging methods.*

| Edging pattern[1] | Log-scaling diameter (in.) | Log length (ft) | Actual log volume[2] ($ft^3$) | Actual lumber yield[3] (bf) | Lumber recovery factor | Component recovery: Flitches (%) | Lumber[4] (%) | Slabs (%) | Edgings (%) | Trims (%) | Chippable residue (%) | Sawdust (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 12[5] | 14.5 | 13.73 | 103.2 | 7.516 | 82.0 | 62.6 | 4.0 | 15.5 | 1.3 | 20.8 | 16.6 |
| 3 | 12 | 14.5 | 13.73 | 119.5 | 8.704 | 82.0 | 72.5 | 4.0 | 5.3 | 1.5 | 10.8 | 16.7 |
| 3/4[6] | 12 | 14.5 | 13.73 | 128.9 | 9.388 | 82.0 | 78.1 | 4.0 | 1.1 | 1.6 | 6.7 | 15.2 |
| 4 | 12 | 14.5 | 13.73 | 133.3 | 9.708 | 82.0 | 81.0 | 4.0 | 0.5 | 1.7 | 6.2 | 12.8 |

[1] As shown in Fig. 1.

[2] Actual log volume was obtained using the following equation: $(1/3)\pi l(r^2 + R^2 + rR)$.

[3] Actual lumber yield was determined on the basis of actual rather than nominal lumber dimensions. The yield based on nominal dimensions is 10% lower.

[4] The percent lumber recovery divided by 100 equals the utilization coefficient.

[5] Diameter at large end equal to 14.32 in.

[6] Part of the lumber processed in this group was edged using Pattern 3 and part was edged using Pattern 4. After edging, lumber was trimmed two inches on each end.

edging products produced, including edgings, edging sawdust, and lumber, using any combination of the six possible edging patterns.

To illustrate the validity of the edging models and their possible usefulness, the models were incorporated into a computerized sawmill simulation program and several logs were computer sawn under a wide range of edging patterns. Results from these initial sawmill computer simulations revealed that the edging models were capable of generating reasonable and realistic edging component data, and that they have the potential for being useful analytical tools in sawmill studies such as fiber balance analysis, and lumber recovery and grade comparisons.

Incorporation of the edging models into sawmill simulation programs presently being developed could represent a significant and important step in the evolution of sawmill simulation. With the increased capability of a more encompassing edging analysis, sawmill simulation could become a more powerful and useful investigative and educational tool to both practitioners and students of the sawmill industry. It is also hoped that the basic approach to edging analysis illustrated in this study, as well as the models developed, can be useful to other investigators studying simulation and analysis of wood processing systems.

## ACKNOWLEDGMENTS

The research described in this paper was partially supported by McIntire-Stennis Project 2494. This article was approved and authorized for publication as Paper No. 6801 in the Journal Series of the Pennsylvania Agricultural Experiment Station.

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