ESTIMATING LOCAL COMPLIANCE IN A BEAM FROM BENDING MEASUREMENTS PART I. COMPUTING "SPAN FUNCTION"

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ABSTRACT

Bending modulus of elasticity measurements have been useful and profitable for decades in the sorting of dimension lumber for its structural quality. Bending and tensile strengths of lumber are known to be correlated with modulus of elasticity. Previous research indicates that bending elasticity on short spans, shorter than can be practically measured with precision, may improve correlation with strength. It is expected, therefore, that the optimal estimation method of the present two-part paper will be applied in the machine stress rating (MSR) process for more accurate sorting of dimension lumber into MSR grades.

Using weighting functions called "span functions," the estimation method processes a sequence of bending measurements from overlapping spans, such as those obtained from equipment for MSR lumber production. A span function is specific to the support configuration of a particular bending span and defines how much the local elastic properties along a beam contribute to a measurement. Intuitively, the local elasticity values of a beam near span center affect the measurement more than values near span ends. Span function defines this effect as a function of position along the bending span. In Part I, a procedure is developed for computing span function of a general bending span configuration. Span functions are graphed for bending spans of a production-line machine used in MSR lumber production and for other bending span configurations. In Part II, use of span functions in optimal estimation of local elasticity is described.

Keywords: Span function, local modulus of elasticity, local compliance, stress-rated, MOE, MEL, MSR, beam.

INTRODUCTION

Bending modulus of elasticity measurements have been useful and profitable for decades in the sorting of dimension lumber for its structural quality. Bending and tensile strengths of lumber are known to be correlated with modulus of elasticity. Previous research indicates that bending elasticity on short spans, shorter than can be practically measured with precision, may improve correlation with strength. It is expected, therefore, that the optimal estimation method of the present two-part paper will be applied in the machine stress rating (MSR) process for more accurate sorting of dimension lumber into MSR grades.

Wood is a highly variable material, and there has been significant historical interest in better determination of local modulus of elasticity. Clearly, the assumption that local modulus of elasticity within a bending span is uniformly the same as measured modulus of elasticity from a bending measurement is made only because of the inability to obtain the local values from bending measurements. An early paper (Kass 1975) discusses some of the precision difficulties involved with bending measurements for short span lengths. Kass described method and laboratory equipment for determining bending values on various span lengths from 203 mm to 610 mm and was able to show evidence of a knot corresponding to minima for short spans that was not evident from longer span data. Other work has shown better correlation with strength when short spans are used (Orosz 1976).

OBJECTIVES

In Part I, the objectives are to define "span function," to develop a general procedure for its computation, and to present examples of useful span functions in graphical form. In Part II, a method is presented that uses a sequence of bending measurements and their corresponding span functions to optimally estimate local elasticity values. While the work has been described as having mathematical complexity[1], the steps included are sufficiently described so that a serious reader should be able to follow the development and duplicate the work.

BACKGROUND

Modulus of elasticity and compliance

Almost all the literature involving bending measurements and the elasticity properties leading to them has involved modulus of elasticity, but the present work uses compliance. From measured modulus of elasticity E_m, measured compliance is $C_m = 1/E_m$. Similarly, from local modulus of elasticity E, local compliance is $C = 1/E$. Compliance, not modulus of elasticity, is used because the derivation shows it is compliance that has a desired convolution relationship useful for the estimation method of Part II.

Because E and cross-sectional moment of inertia "I" appear as a product in flexural loading equations (Higdon et al. 1960), the results may be generalized. The elastic property definitions are modified to include moment of inertia so that measured compliance becomes $C_m = 1/(EI)_m$, where $(EI)_m$ is the bending measurement of the EI product, and local compliance $C = 1/(EI)$ is defined from the local values E and I. Defining a variable local compliance as the reciprocal of the EI product allows for either E, I, or both E and I to be variable. In much of the prior work using flexural loading equations, computational simplification is achieved by neglecting variations in E and I. The present contribution is in dealing with the situation when these variations are not negligible.

Formulas for computing C_m are derived from equations relating force, shear, moment, slope, and deflection, each (except force) as an integral of the preceding quantity (Higdon et al. 1960). The slope equation includes in the integrand, the local factors E and I in the denominator, as well as bending moment in the numerator. As in Higdon et al. (1960), the amount of bending is assumed small, and plane cross-sections of the beam before bending are assumed to remain plane after bending. The implication is that shear deflections in the beam are negligible. This assumption is reasonable for bending spans used in high-speed production-line machines wherein the bending span length to beam depth ratio is on the order of 32 and where beam ends usually extend past the bending span ends. For bending configurations having much smaller length-to-depth ratios, this assumption may be questionable.

Review of previous work

For simply-supported, center-loaded bending spans, Bechtel (1985) derived a weighting function, illustrated here as Fig. 1, showing the effect on bending compliance measurements contributed by the local compliance at each point along

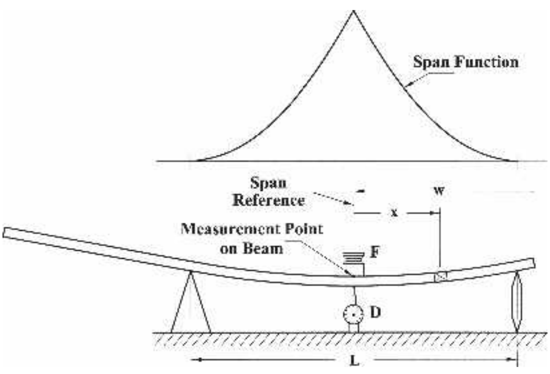

FIG. 1. Beam simply-supported and center-loaded on a bending span covering only a part of the beam length. The applicable span function is shown aligned with the bending span and illustrates how much the local compliances contribute to measured compliance.

[1]An overview is available for those wishing to avoid the mathematical details (Bechtel et al. 2007).

a beam. This cusp-shaped function is a combination of two quadratic functions, which, as expected from intuition, is greatest at span center and least at span ends. The name span function is applied to this as well as to weighting functions for other types of bending spans.

MEASURED COMPLIANCE

There are two abscissa coordinates defined in Fig. 1. A beam coordinate identifies a point on the beam relative to its leading end, here taken as the right end of the beam. A beam coordinate is distance along the beam positive to the left from the beam's right end. A span coordinate identifies a point of the bending span relative to a span reference, which in Fig. 1 is the center of the span. A span coordinate measures distance positive to the right from the span reference. A "measurement point" is a point on the beam that coincides with the bending span reference. As the beam in Fig. 1 moves to the right, the measurement point moves leftward along the beam.

Measured compliance $C_m(w)$ is the compliance measurement when the measurement point on the beam is at distance w from its leading end. It is given by:

$$C_m(w) = \frac{D(w)}{F(w)} \tag{1}$$

where D(w) is a function of the support deflections, and F(w) is a function of the support forces. Analysis for the configuration of Fig. 1 identifies three supports, one at each end of the bending span and one in the center. In Fig. 1, D(w) is the measured incremental deflection at the center support as a result of the applied load weight F(w). F(w) will not change with w in this case, but D(w) will change as the beam moves to the right (increasing w). Given a span function h(x), where x is a span coordinate, measured compliance may be written as a convolution integral:

$$C_m(w) = \int_{-L/2}^{L/2} C(w-x)h(x)dx \tag{2}$$

Equation (2) weights the local compliance C(w-x) of the beam at distance w-x from its leading end with the weight h(x)dx and sums all such weighted compliances over the extent of the bending span to arrive at a measured compliance $C_m(w)$ at the measurement point. The limits of integration could equally well have covered the length of the beam, but have been limited in Eq. (2) because it is recognized that the span function h(x) is zero outside the span from $-L/2$ to $L/2$. A property of span functions may be obtained from Eq. (2). If the local compliance is uniform, that is, $C(w\text{-}x) = C_o$, then certainly, the measured compliance should also be C_o. By removing the constant C_o from the integral, it is seen that the integral of h(x) is one; that is, the weights add to one.

SPAN FUNCTION DETERMINATION AS A DERIVATIVE

Consider the change in measured compliance if an infinitesimally small impulse of compliance is added to the compliance function at distance ξ from the leading end of the beam. This can be modeled in Eq. (2) by adding a compliance impulse of weight b at position ξ and letting b approach zero. The local compliance $C_{b,\xi}(u)$ as a function of the beam coordinate u after adding the impulse is:

$$C_{b,\xi}(u) = C(u) + b\delta(u-\xi) \tag{3}$$

C(u) is the local compliance before adding the impulse, and δ is a Dirac delta function defined by:

$$\delta(u-\xi) = \begin{cases} \infty, & \text{if } u = \xi \\ 0, & \text{otherwise} \end{cases} \quad \text{such that}$$

$$\begin{cases} \int_{a1}^{a2} \delta(u-\xi)\,du = 1, \text{ if } \xi \in [a1,a2] \\ \int_{a1}^{a2} \delta(u-\xi)\,du = 0, \text{ if } \xi \notin [a1,a2] \end{cases} \tag{4}$$

The arbitrary limits of integration a1 and a2 satisfy a1< a2. Substituting local compliance function $C_{b,\xi}(u)$ from (3) into the convolution integral of (2) gives:

$$
\begin{aligned}
&C_m(w,b,\xi)\\
&\quad = \int_{-L/2}^{L/2} C_{b,\xi}(w-x)h(x)dx\\
&\quad = \int_{-L/2}^{L/2} (C(w-x) + b\delta(w-x-\xi))\, h(x)dx\\
&\quad = C_m(w) + b\int_{-L/2}^{L/2} \delta(w-x-\xi)h(x)dx\\
&\quad = C_m(w) + bh(w-\xi)\\
&\quad = C_m(w) + bh(x) = C_m(b,x),\\
&\quad \text{where } x = w-\xi
\end{aligned}
\tag{5}
$$

In Eq. (5), first, x is a dummy variable of integration. After it is removed by integration, x is used again to replace w-ξ and should be considered as the distance measured, positive to the right, from the span reference to the impulse. Notation in Eq. (5) for measured compliance $C_m(w,b,\xi)$ first references position w of the measurement point and impulse position ξ as beam coordinates. Then it is simplified to $C_m(b,x)$ to indicate position x = w-ξ of the impulse as a span coordinate. $C_m(b,x)$ should be interpreted as the measured compliance when the beam local compliance is C(w) plus a compliance impulse of weight b at distance x to the right of the span reference. From these definitions, it is seen that $C_m(b,x)|_{b=0} = C_m(0,x) = C_m(w)$. Rearranging the result from (Eq. 5) and letting b go to zero:

$$
h(x) = \lim_{b\to 0}\left[\frac{C_m(b,x) - C_m(0,x)}{b}\right] = \frac{\partial C_m(b,x)}{\partial b}\Big|_{b=0} \tag{6}
$$

Thus, the span function at x is given as the partial derivative of measured compliance $C_m(b,x)$ with respect to impulse weight b, evaluated at b = 0. By exploring this derivative over x, the span function is obtained.

Computed versus measured span function

The span function h(x) may be computed from Eq. (6) as will be demonstrated. However, it can, in principle, also be obtained by measurement. Measured compliance without impulse is subtracted from measured compliance with impulse, this being repeated for impulse positions along the bending span. Then the difference function is scaled so that, as a function of impulse position, it integrates to one. A compliance impulse can be approximated in the beam, e.g. by drilling holes in the beam at the desired location or by a saw kerf. But there are problems with this approach. First, as the impulse becomes suitably small, noise riding on the measurement is large relative to the difference signal. Second, there is the difficulty of maintaining a given local compliance function C(u) while changing the relative position of the impulse; however, this problem is eliminated for a beam having uniform compliance (except for the impulse). Obtaining the impulse weight is not a problem, because scaling the difference function as described is equivalent to dividing the difference function by the impulse weight.

For most cases where support conditions are known, it is simpler and more accurate to compute the "measured" compliance. Rather than start with just any background local compliance function C(u) as in Eq. (3), it is much more convenient to use a uniform compliance C_o as the background compliance. Usually, the partial derivative of Eq. (6) will be computed from the limiting operation of Eq. (6).

Approach to computing span function

The approach consists of defining a test function of local compliance $C_t(u)$ comprising a constant C_o plus an impulse of weight b at position ξ from the leading end of the board:

$$
C_t(u) = C_o + b\delta(u-\xi) \tag{7}
$$

For this test function of local compliance, the difference of measured compliance with and without impulse is computed and divided by the weight b, and the limit is taken as b approaches zero. These steps, per Eq. (6), are repeated for all desired values of the independent variable x.

The only remaining difficulty is in the computation of the measured compliance with the impulse included in the local compliance function. Without this impulse, measured compliance is simply the uniform value C_o.

Computational details for general span configuration

Consider the general bending span configuration of Fig. 2, where an arbitrary number n of supports are located at $x_1, x_2, \ldots, x_n$ relative to the span reference **5**. At these supports, the forces and deflections are $F_1, F_2, \ldots, F_n$ and $D_1, D_2, \ldots, D_n$ respectively with positive sense upward. Using methods from Higdon et al. (1960), the following system of equations may be written:

Force

$$F(x) = \sum_{i=1}^{n} F_i \delta(x - x_i)$$

Shear

$$V(x) = \sum_{i=1}^{n} F_i \int_{x_1}^{x} \delta(u - x_i)du = \sum_{i=1}^{n} F_i U(x - x_i)$$

Moment

$$M(x) = \sum_{i=1}^{n} F_i \int_{x_1}^{x} U(u - x_i)du = \sum_{i=1}^{n} F_i r(x - x_i) \qquad (8)$$

Slope

$$S(x) = S_1 + \int_{x_1}^{x} C(w - u)M(u)du = S_1 + \sum_{i=1}^{n} F_i \int_{x_1}^{x} C(w - u)\, r(u - x_i)du$$

Deflection

$$D(x) = D_1 + S_1 (x - x_1) + \sum_{i=1}^{n} F_i \int_{x_1}^{x}\int_{x_1}^{v} C(w - u)r(u - x_i)du\, dv$$

In Eqs. (8), effects of distributed loads (e.g. from mass of beam) are neglected. Shear and moment to the left of the first support in Fig. 2 or to the right of the last support have been assumed to cause negligible effect. This assumption can be made reasonable by defining additional supports to reduce these effects. The clamp roller supports on some production-line machines (e.g. see Fig. 3) make this assumption reasonable. S_1 and D_1 are respectively the beam slope and deflection at the first support x_1. Concentrated forces at the supports are introduced via impulse functions. The symbols U and r represent the unit step and the unit ramp defined by:

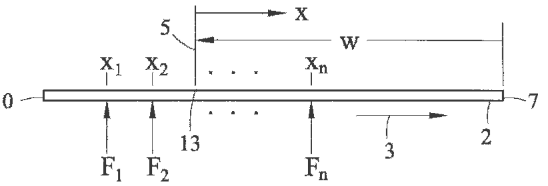

FIG. 2. Beam on a bending span having a general system of n supports at locations x_1 through x_n. Concentrated forces at the supports and deflections at the supports have positive sense upward. Beam coordinate "w" increases leftward from leading end **7**. Measurement point **13** on the beam is shown aligned with a span reference **5**. Span coordinate "x" increases rightward from the span reference.

Unit step function $$U(x - x_i) = \begin{cases} 0, & \text{if } x < x_i \\ 1, & \text{if } x \geq x_i \end{cases} \qquad (9)$$

Unit ramp function $$r(x - x_i) = \begin{cases} 0, & \text{if } x < x_i \\ x - x_i, & \text{if } x \geq x_i \end{cases}$$

The last of Eqs. (8) is evaluated at each of the n-1 support locations $x_2, \ldots, x_n$, and the notation is simplified from $D(x_j)$ to D_j:

$$D_j = D_1 + S_1(x_j - x_1) + \sum_{i=1}^{j-1} F_i \int_{x_1}^{x_j}\int_{x_1}^{v} C(w - u)r(u - x_i)du\, dv, \quad \text{for} \quad j = 2, \ldots, n \qquad (10)$$

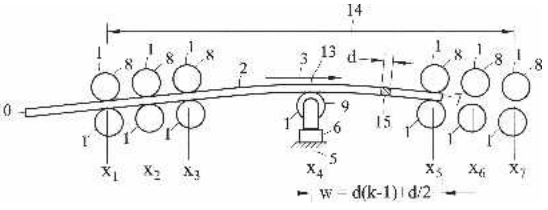

FIG. 3. A production-line configuration of seven supports is illustrated with a beam **2** engaging only five of them.

The upper limit of the sum was changed to j-1 because for higher indices, the ramp function is zero (in the integrand $u \leq v \leq x_j$, and hence the argument $u - x_i \leq 0$ for $i \geq j$). The double integral may be reduced to a single integral by changing the order of integration:

$$\begin{aligned} D_j &= D_1 + S_1(x_j - x_1) \\ &\quad + \sum_{i=1}^{j-1} F_i \int_{x_1}^{x_j} C(w-u) r(u-x_i) \int_u^{x_j} dv\, du \\ &= D_1 + S_1(x_j - x_1) \\ &\quad + \sum_{i=1}^{j-1} F_i \int_{x_i}^{x_j} C(w-u)\,(x_j-u)(u-x_i)du, \\ &\qquad \text{for} \quad j = 2, \ldots, n \end{aligned} \tag{11}$$

The change in order of integration and in integration limits from Eq. (10) to the first expression of Eq. (11) is best understood by sketching the triangular region defined by these limits in the two-dimensional (u,v) plane of integration. Because the ramp function is zero for $u < x_i$, the lower integration limit in the second line of Eq. (11) is changed from x_1 to x_i, and then $r(u\text{-}x_i) = (u\text{-}x_i)$. The notation is further simplified by defining:

$$I_{i,j} = \int_{x_i}^{x_j} C(w-u)(x_j-u)(u-x_i)du,\ j > i \tag{12}$$

and, without loss of generality, $D_1 = 0$, so that the n-1 Eqs. (11) may be written:

$$D_j = S_1(x_j - x_1) + \sum_{i=1}^{j-1} F_i I_{i,j}, \quad j = 2, \ldots, n \tag{13}$$

The introduction of two additional equations summing forces to zero in the vertical direction and summing moments to zero in the plane of bending about the last support (system is not accelerating either in translation or in rotation), allows a system of n + 1 equations to be written in matrix form as:

$$W\underline{F} = \underline{D} \tag{14}$$

where:

$$W = \begin{bmatrix} I_{1,2} & 0 & \ldots & 0 & 0 & x_2 - x_1 \\ I_{1,3} & I_{2,3} & \ddots & \vdots & \vdots & x_3 - x_1 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ I_{1,n} & I_{2,n} & \ldots & I_{n-1,n} & 0 & x_n - x_1 \\ 1 & 1 & \ldots & 1 & 1 & 0 \\ x_n - x_1 & x_n - x_2 & \ldots & x_n - x_{n-1} & 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n-1} \\ F_n \\ S_1 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} D_2 \\ D_3 \\ \vdots \\ D_n \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

Inserting the test function $C_t(u)$ into the integral of Eq. (12) gives:

$$\begin{aligned} I_{i,j} &= \int_{x_i}^{x_j} (C_o + b\delta(w-u-\xi))(x_j-u)(u-x_i)du \\ &= C_o \int_{x_i}^{x_j} (x_j-u)(u-x_i)\,du \\ &\quad + b\int_{x_i}^{x_j} \delta(x-u)\,(x_j-u)(u-x_i)\,du \\ &= C_o \frac{(x_j-x_i)^3}{6} + b d_{i,j}(x) \end{aligned} \tag{16}$$

where $x = w\text{-}\xi$ is the position of the impulse in the span coordinate system and $d_{i,j}(x)$ is defined by:

$$d_{i,j}(x) = \begin{cases} (x_j - x)(x - x_i), & \text{if } x_i < x < x_j \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Using Eq. (17) in (16) and then that result in matrix W from Eqs. (15), W may be written as the sum of two parts:

$$W = W_o + bW_d\,(x) \tag{18}$$

where W_o and $W_d(x)$ are:

$$
W_o = \begin{bmatrix}
C_o\,(x_2-x_1)^3/6 & 0 & \cdots & 0 & 0 & x_2-x_1 \\
C_o\,(x_3-x_1)^3/6 & C_o\,(x_3-x_2)^3/6 & \ddots & \vdots & \vdots & x_3-x_1 \\
\vdots & \vdots & \ddots & 0 & \vdots & \vdots \\
C_o\,(x_n-x_1)^3/6 & C_o\,(x_n-x_2)^3/6 & \cdots & C_o\,(x_n-x_{n-1})^3/6 & 0 & x_n-x_1 \\
1 & 1 & \cdots & 1 & 1 & 0 \\
x_n-x_1 & x_n-x_2 & \cdots & x_n-x_{n-1} & 0 & 0
\end{bmatrix} \tag{19}
$$

$$
W_d(x) = \begin{bmatrix}
d_{1,2}(x) & 0 & \cdots & 0 & 0 & 0 \\
d_{1,3}(x) & d_{2,3}(x) & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 0 & \vdots & \vdots \\
d_{1,n}(x) & d_{2,n}\,(x) & \cdots & d_{n-1,n}\,(x) & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \tag{20}
$$

For example, if x happens to be located between the second and third supports:

$$
W_d(x) = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
(x_3-x)(x-x_1) & (x_3-x)(x-x_2) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(x_n-x)(x-x_1) & (x_n-x)(x-x_2) & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \text{ if } x_2 < x < x_3 \tag{21}
$$

SPECIFIC USEFUL BENDING SPAN CONFIGURATIONS

Production-line machine

In each bending section (see Fig. 3) of one type of machine commonly used to produce machine stress-rated lumber, a beam fully engaged will contact seven supports (n = 7). The supports are defined by rollers, and the upper six rollers marked **8** are motor driven to propel the beam **2** in direction **3**. Six lower rollers marked **1** clamp the beam firmly against the upper rollers to define beam deflections at six support points. The roller marked **9** fixes the deflection at the 4th support where the load is measured by load cell **6**. Measured compliance is given by:

$$
C_m\,(b, x) = \frac{K}{F_4} = \frac{K}{\{\underline{F}\}_4} = \frac{K}{\{W^{-1}\underline{D}\}_4} \tag{22}
$$

The calibration constant K is adjusted so that for a beam having uniform local compliance C_o, and thus impulse weight b = 0 and matrix W = W_o, the measured compliance is $C_m(0,x) = C_o$. The subscripted brace notation, for example $\{\underline{F}\}_4$ introduced in Eq. (22), denotes the fourth component of the vector enclosed in the braces.

In the following derivation, $W_d(x)$ is written as W_d, its dependence on x being understood. Use is made of the fact that an inverse matrix $(I+bA)^{-1}$ may be replaced with I-bA for very small scalar b, where the identity matrix I agrees in size with matrix A. Similarly, the fraction 1/(1-ba), where a is a scalar, may be replaced with 1+ba when b is very small. The span function h(x) is computed in closed form per Eq. (6) as detailed by the following steps.

$$
\begin{aligned}
h(x) &= \lim_{b\to 0}\left(\frac{C_m(b,x)-C_o}{b}\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{\underline{F}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{W^{-1}\underline{D}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{(W_o+bW_d)^{-1}\underline{D}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{(I+bW_o^{-1}W_d)^{-1}W_o^{-1}\underline{D}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{(I-bW_o^{-1}W_d)W_o^{-1}\underline{D}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{1}{b}\left(\frac{K}{\{W_o^{-1}\underline{D}\}_4-b\{W_o^{-1}W_dW_o^{-1}\underline{D}\}_4}-C_o\right)\\
&= \lim_{b\to 0}\frac{C_o}{b}\left(\frac{1}{1-b\dfrac{\{W_o^{-1}W_dW_o^{-1}\underline{D}\}_4}{\{W_o^{-1}\underline{D}\}_4}}-1\right)\\
&= \lim_{b\to 0}\frac{C_o}{b}\left(1+b\frac{\{W_o^{-1}W_dW_o^{-1}\underline{D}\}_4}{\{W_o^{-1}\underline{D}\}_4}-1\right)\\
&= C_o\frac{\{W_o^{-1}W_dW_o^{-1}\underline{D}\}_4}{\{W_o^{-1}\underline{D}\}_4}
\end{aligned}
\tag{23}
$$

It is possible, although not necessary, to avoid entering a value C_o for the arbitrary constant compliance. By multiplying the second to last of the equations in the matrix system of Eq. (14) by C_o so that the 1's in W_o become C_o, it can be shown that the constant compliance C_o cancels in the span function computation. This involves partitioning the W_o matrix into four component matrices, only one having C_o as a factor. In the upper-left component matrix of the partitioned inverse of W_o, the constant compliance C_o will appear only as the multiplier $1/C_o$. This component matrix is the only one needed in the computation of span function because of the zeros in vector $\underline{D}$ and matrix W_d. Because the inverse matrix W_o^{-1} appears as a factor twice in the numerator for h(x) of Eq. (23) and once in the denominator, all factors of C_o cancel in the computation. These details can be used to write the span function without specifying C_o. The result, reinserting notation showing dependence of W_d on x, is:

$$
h(x) = \frac{\{W_{\bullet}^{-1}\,W_d\,(x)W_{\bullet}^{-1}\underline{D}\}_4}{\{W_{\bullet}^{-1}\,\underline{D}\}_4} \tag{24}
$$

where $W_{\bullet}$ is the matrix of Eq. (19), but with C_o removed or just set to one.

$$
W_{\bullet} = W_o\big|_{C_o} = 1 \tag{25}
$$

In the configuration of Fig. 3, there are actually 5 different span functions applicable for beams longer than the distance between first and last supports. As a beam enters the support system, useable measurements are possible when the beam's leading end, marked **7** in Fig. 3, first engages support five as shown. From there until it engages support six, a five support system is applicable, and the span function for it is computed similarly to the one above. And then, until the leading end engages support seven, a six support system is applicable. Thus, first 5, then 6, then 7, then 6, and then 5 supports are applicable. The last two are obtained as the trailing end **10** of the beam disengages the first support and then the second. Figure 4 illustrates all five of these span functions **31**, **32**, **33**, **34**, and **35** in sequence as the beam progresses rightward engaging respectively five, six, seven, six and five supports. The differences in span functions among the 6 and 7 support spans are not apparent on the scale of Fig. 4. Differences among the 5 and 7 support cases are readily apparent.

Bending proof load testers used to measure bending compliance

In North America two different configurations are commonly used in bending proof load testers for off-line quality control of bending stiffness where, usually, bending is in the stiff direction (edgewise). Figure 5 illustrates both configurations. There are four equally spaced supports that provide loading of a beam **2**. Loads are applied equally in one direction at the inner two loading supports with equal reaction forces in the opposite direction at the outer two supports. However, in the first configuration, beam deflec-

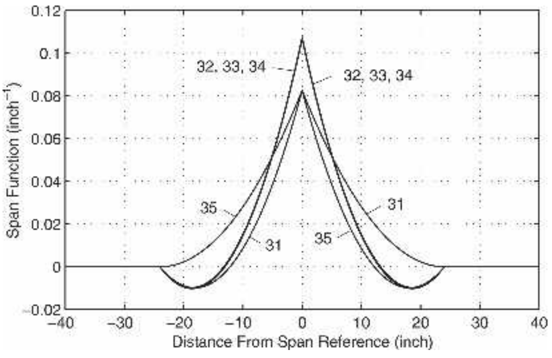

FIG. 4. Span functions for the five applicable spans of Fig. 3 for beams longer than x_7-x_1. The sequence of 5, 6, 7, 6, and then 5 supports engaging the beam as the beam progresses to the right through the system has, respectively, the span functions labeled **31**, **32**, **33**, **34**, and **35**. The span functions for the 6- and 7-support cases are indistinguishable from one another on the scale of this graph. The 5-support span functions **31** and **35** are clearly different from the others and from each other.

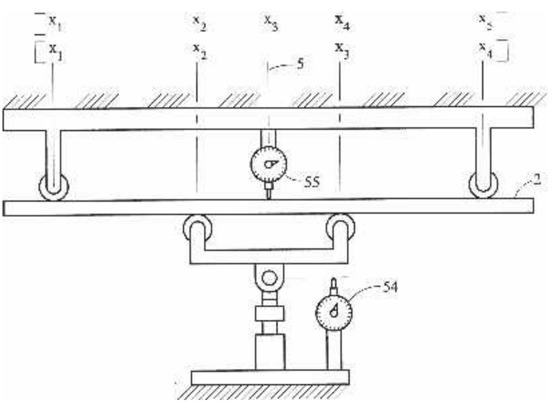

FIG. 5. Mechanical schematic for two commonly used bending proof load testing configurations. The first measures deflection at **54** as the average of the loading support deflections. The second measures deflection **55** at span center.

tion is measured **54** as the average of the support deflection at the inner two loading support points, while in the second; deflection is measured **55** at span center. A similar process is used as for the production-line machine to compute span function. For the first configuration, the computed measured compliance C_m is from a ratio of deflection to force where the deflection is the average of the deflections at the inner loading supports and the force is the sum of the forces at those supports. In this case the brace notation (see Eq. (22) and discussion) is redefined as a combination of the forces and deflections at the inner loading supports. For analysis of the second configuration, a zero-force center support point is introduced as an additional support between the two inner supports, and the deflection at center span is measured at this support. A five support system is solved where the center support force is specified as zero. Except for this additional complication, a similar process is used. Span functions for both of these two configurations of proof testers are illustrated in Fig. 6. Full details of these computations may be found in a United States patent specification (Bechtel et al. 2006). From Fig. 6, it is clear that span functions for these machine configurations are not identical even though bending moments in a tested beam are. Compliances along the center third of the span contribute equally in the first case, marked **40** in Fig. 6, but not in the second, marked **41**. This difference could be important to those specifying procedures for bending tests of beams, e.g. (ASTM 2005).

CONCLUSIONS

Span function is defined as a weighting function that describes how local compliances in a beam affect a bending measurement of compliance. The span function depends on both the position of a local compliance relative to the

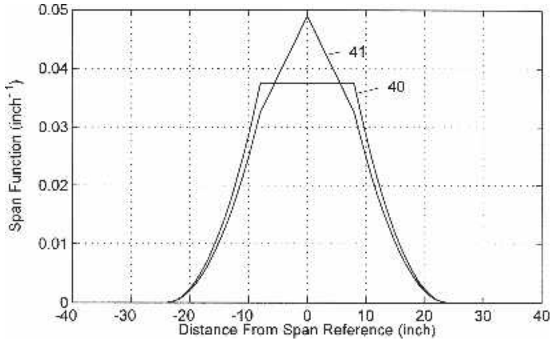

FIG. 6. Span functions for the two configurations of Fig. 5. Span function for deflection measurement **54** in Fig. 5 is shown as curve **40**. Span function for deflection measurement **55** in Fig. 5 is shown as curve **41**.

bending span and on the bending support configuration used. A span function may be evaluated as a partial derivative of measured compliance with respect to impulse weight when an impulse of compliance is added to the local compliance function. Examples for several useful bending span configurations are given. Span functions as computed here in Part I are useful to the method of Part II for optimally estimating local compliance values.

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