ANALYSIS OF WOOD CANTILEVER LOADED AT FREE END

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ABSTRACT

A wood cantilever loaded at the free end was analyzed using the anisotropic elasticity theory. This report presents a two-dimensional numerical example of a Sitka spruce cantilever in the longitudinal–radial plane. When the grain slope is zero, i.e., the beam axis coincides with the longitudinal axis of wood, the stresses in the beam and the deflection of the beam are the same as those for an isotropic beam; when the grain slope is different from zero, the stresses and the deflection can increase significantly.

Keywords: Bending stress, cantilever, deflection, orthotropic material, shear stress, wood.

INTRODUCTION

Wood may be described as an orthotropic material with independent mechanical properties in the directions of three mutually perpendicular axes: longitudinal (L), radial (R), and tangential (T). These are called the principal material axes, and the mechanical properties referred to them are the engineering constants. The material axes and the geometrical axes used to describe a rectangular structural member do not usually coincide. According to Hoyle, Jr. (1982), the angle between a material axis and an adjacent geometrical axis can be as much as $\pm 15°$. The mechanical properties referred to the geometrical axes are called the transformed engineering constants. In a two-dimensional situation, the relations between transformed engineering constants and engineering constants, between transformed stiffness and principal stiffness, and between transformed compliance and principal compliance are well documented (Jones 1975; Tsai and Hahn 1980).

Kilic et al. (2001) analyzed the effects of shear on the deflection of an orthotropic cantilever loaded either uniformly or by a single force at the free end. In their analysis, they referred the shear effects to the geometrical axes only, which is of limited interest in the design of a wood structural member. In wood engineering, all the independent mechanical properties are referred to the material axes.

In this study, we investigated the stress distributions and the deflection curves of an orthotropic cantilever loaded at the free end using the anisotropic elasticity theory by Lekhnitskii (1968). The effects of shear on deflection for several values of grain slope referred to the material axes are analyzed. Numerical results are presented for a Sitka spruce (*Picea sitchensis* (Bong.) Carr.) cantilever beam.

GENERAL ANISOTROPIC ELASTICITY

Let axes 1 and 2 define the principal material plane, with axis 1 in the grain direction and axis 2 in the radial direction. The geometrical axes x and y are located at the free end of the beam, with axis x at an angle θ from axis 1 (Fig. 1). Angle θ is called the grain slope. The stress/strain relations in anisotropic elasticity theory are shown in Eqs. (1) (Tsai and Hahn 1980):

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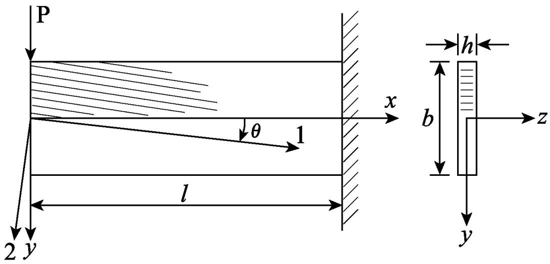

FIG. 1. Orthotropic cantilever subjected to single load. x and y are geometrical axes; 1 and 2 are material axes.

$$\varepsilon_x = S_{xx}\sigma_x + S_{xy}\sigma_y + S_{xs}\tau_{xy} \tag{1a}$$

$$\varepsilon_y = S_{yx}\sigma_x + S_{yy}\sigma_y + S_{ys}\tau_{xy} \tag{1b}$$

$$\gamma_{xy} = S_{sx}\sigma_x + S_{sy}\sigma_y + S_{ss}\tau_{xy} \tag{1c}$$

where the transformed compliances S_{ij} can be given as

$$\begin{aligned}
S_{xx} &= m^4S_{11} + n^4S_{22} + 2m^2n^2S_{12} + m^2n^2S_{66}\\
S_{yy} &= n^4S_{11} + m^4S_{22} + 2m^2n^2S_{12} + m^2n^2S_{66}\\
S_{xy} &= S_{yx} = m^2n^2S_{11}\\
&\quad + m^2n^2S_{22} + (m^4 + n^4)S_{12} - m^2n^2S_{66}\\
S_{xs} &= S_{sx}\\
&= 2m^3nS_{11} - 2mn^3S_{22} + 2(mn^3 - m^3n)S_{12}\\
&\quad + (mn^3 - m^3n)S_{66}\\
S_{ys} &= S_{sy}\\
&= 2mn^3S_{11} - 2m^3nS_{22} + 2(m^3n - mn^3)S_{12}\\
&\quad + (m^3n - mn^3)S_{66}\\
S_{ss} &= 4m^2n^2S_{11} + 4m^2n^2S_{22} - 8m^2n^2S_{12}\\
&\quad + (m^2 - n^2)^2S_{66}
\end{aligned} \tag{2}$$

with $m = \cos\theta$ and $n = \sin\theta$ and the principal compliances

$$S_{11} = \frac{1}{E_1} \qquad S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\upsilon_{12}}{E_1} = S_{21} = -\frac{\upsilon_{21}}{E_2} \qquad S_{66} = \frac{1}{G_{12}} \tag{3}$$

In addition, the strain/displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{4a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{4b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4c}$$

in which u and v are displacements in x and y directions, respectively.

DERIVATION OF DEFLECTION CURVES

Stress components at a point (x, y) of a cantilever subjected to a single load P at the free end are (Lekhnitskii 1968)

$$\begin{aligned}
\sigma_x &= -\frac{P}{I}xy + \frac{PS_{xs}}{IS_{xx}}\left(\frac{b^2}{12} - y^2\right)\\
\sigma_y &= 0\\
\tau_{xy} &= -\frac{P}{2I}\left(\frac{b^2}{4} - y^2\right)
\end{aligned} \tag{5}$$

where $I = hb^3/12$; h and b are the width and height of beam cross section in Fig. 1. From Eqs. (1a), (5), and (4a), we obtain

$$u = -\frac{P}{2I}\left[S_{xx}x^2y + \frac{S_{xs}}{12}(b^2 + 12y^2)x\right] + f(y) \tag{6}$$

From Eqs. (1b), (5), and (4b), it follows

$$v = \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y - S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] + g(x) \tag{7}$$

Then, from Eqs. (1c), (5) and (4c), we obtain, by means of Eqs. (6) and (7),

$$\frac{PS_{xx}}{2I}x^2 + \frac{P}{I}\left(\frac{S_{ss} + S_{xy}}{2} - \frac{S_{xs}}{S_{xx}}\right)y^2 + \frac{Pb^2}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) - f'(y) - g'(x) = 0 \tag{8}$$

In Eq. (8), some terms are functions of x only, some are functions of y only, and one is independent of both x and y. Denoting these groups by $F(x)$, $G(y)$, and K, we have

$$F(x) = \frac{PS_{xx}}{2I}x^2 - g'(x)$$

$$G(y) = \frac{P}{I}\left(\frac{S_{ss} + S_{xy}}{2} - \frac{S_{xs}^2}{S_{xx}}\right)y^2 - f'(y)$$

$$K = -\frac{Pb^2}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right)$$

and Eq. (8) may be written

$$F(x) + G(y) = K$$

Since K is independent of x and y, we must set $F(x)$ equal to some constant d and $G(y)$ some constant e. Thus,

$$d + e = -\frac{Pb^2}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) \qquad (9)$$

and

$$\frac{dg(x)}{dx} = \frac{PS_{xx}}{2I}x^2 - d$$

$$\frac{df(y)}{dy} = \frac{P}{I}\left(\frac{S_{ss} + S_{xy}}{2} - \frac{S_{xs}^2}{S_{xx}}\right)y^2 - e$$

Functions $g(x)$ and $f(y)$ are then

$$g(x) = \frac{PS_{xx}}{6I}x^3 - dx + j$$

$$f(y) = \frac{P}{3I}\left(\frac{S_{ss} + S_{xy}}{2} - \frac{S_{xs}^2}{S_{xx}}\right)y^3 - ey + k$$

Substituting in Eqs. (6) and (7), we find

$$u = -\frac{P}{2I}\left[S_{xx}x^2y + \frac{S_{xs}}{12}(b^2 + 12y^2)x\right] + \frac{P}{3I}\left(\frac{S_{ss} + S_{xy}}{2} - \frac{S_{xs}^2}{S_{xx}}\right)y^3 - ey + k \qquad (10)$$

$$v = \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y - S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] + \frac{PS_{xx}}{6I}x^3 - dx + j \qquad (11)$$

The constants d, e, k, and j may now be determined from Eq. (9) and from the three conditions of constraint that are necessary to prevent the beam from moving as a rigid body in the xy-plane. Assuming that u and v are zero for $x = l$, $y = 0$, we find from Eqs. (10) and (11),

$$j = dl - \frac{PS_{xx}}{6I}l^3 \qquad k = \frac{PS_{xs}b^2l}{24I}$$

For determining the constant d in Eq. (11), we must use the third condition of constraint to eliminate the possibility of rotation of the beam in the xy-plane about the center of the fixed end (Timoshenko and Goodier 1951). Two possible constraining conditions are considered:

(1) When an element of the axis of the beam is fixed at the fixed end, we have

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \qquad (12)$$

We obtain from Eq. (11)

$$d = \frac{PS_{xx}}{2I}l^2$$

and Eq. (11) becomes

$$v = \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{yx})y - S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] + \frac{PS_{xx}}{2I}\left[\frac{(x^3 - l^3)}{3} - (x - l)l^2\right] \qquad (13)$$

The deflection curve is obtained by substituting $y = 0$ into Eq. (13). Then,

$$v_{y=0} = \frac{PS_{xx}}{2I}\left[\frac{(x^3 - l^3)}{3} - (x - l)l^2\right] \qquad (14)$$

At the free end,

$$v_{\substack{x=0\\y=0}} = \frac{PS_{xx}l^3}{3I} \qquad (15)$$

(2) When a vertical element at the fixed end is fixed, we have

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \tag{16}$$

From Eq. (10) we obtain

$$e = -\frac{PS_{xx}l^2}{2I}$$

The constant d in Eq. (11) is then obtained from Eq. (9)

$$d = -\frac{Pb^2}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) + \frac{PS_{xx}l^2}{2I}$$

and Eq. (11) becomes

$$\begin{aligned} v = {} & \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y - S_{xy}xy^2 \right. \\ & \left. + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] \\ & + \frac{PS_{xx}}{6I}x^3 + \frac{P}{2I}\left[\frac{b^2}{2}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) - S_{xx}l^2\right]x \\ & + \frac{Pl}{I}\left[\frac{S_{xx}l^2}{3} - \frac{b^2}{4}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right)\right] \end{aligned} \tag{17}$$

The deflection curve is obtained from Eq. (17) with $y = 0$.

$$\begin{aligned} v_{y=0} = {} & \frac{PS_{xx}}{6I}x^3 + \frac{P}{2I}\left[\frac{b^2}{2}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) - S_{xx}l^2\right]x \\ & + \frac{Pl}{I}\left[\frac{S_{xx}l^2}{3} - \frac{b^2}{4}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right)\right] \end{aligned} \tag{18}$$

At the free end,

$$v_{\substack{x=0\\y=0}} = \frac{PS_{xx}l^3}{3I} - \frac{Pb^2l}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) \tag{19}$$

which is the corrected form of the resulting equation obtained by Kilic et al. (2001). Note the first term is identical to Eq. (15).

SHEAR EFFECTS ON DEFLECTION

For an isotropic material, the first term in Eq. (19) is due to flexural and the second term to shear (Timoshenko and Goodier 1951). This approach was also adopted by Kilic et al. (2001). For an orthotropic material, however, that is only true when $\theta = 0$. For $\theta \neq 0$, S_{xx} in the first term as well as S_{xs} and S_{ss} are all functions of the principal compliances, as shown in Eqs. (2) and (3). After the transformed compliances are replaced by the principal compliances, it is appropriate to separate the terms containing S_{66} from those that do not in Eq. (19) to study the effects of shear on deflection based on the consideration of mechanical properties. For this separation, the only term that requires special attention is the ratio S_{xs}^2/S_{xx}.

For abbreviation, we may write from Eq. (2)

$$S_{xs} = \alpha + \beta S_{66}$$

where

$$\alpha = 2m^3nS_{11} - 2mn^3S_{22} + 2(mn^3 - m^3n)S_{12}$$

$$\beta = mn^3 - m^3n$$

and

$$S_{xx} = \gamma + \delta S_{66}$$

where

$$\gamma = m^4S_{11} + n^4S_{22} + 2m^2n^2S_{12} \qquad \delta = m^2n^2$$

We then have

$$\begin{aligned} \frac{S_{xs}^2}{S_{xx}} &= \frac{(\alpha + \beta S_{66})^2}{\gamma + \delta S_{66}} = \frac{\alpha^2\left(1 + \dfrac{\beta}{\alpha}S_{66}\right)^2}{\gamma\left(1 + \dfrac{\delta}{\gamma}S_{66}\right)} \\ &= \frac{\alpha^2}{\gamma}\frac{(1 + \mathrm{X})}{(1 + \mathrm{Y})} \\ &= \frac{\alpha^2}{\gamma}[1 + (\mathrm{X} - \mathrm{Y}) \\ &\qquad \times (1 - \mathrm{Y} + \mathrm{Y}^2 - \mathrm{Y}^3 + \cdots)] \\ &= \frac{\alpha^2}{\gamma} + \frac{\alpha^2}{\gamma}(\mathrm{X} - \mathrm{Y}) \\ &\qquad \times (1 - \mathrm{Y} + \mathrm{Y}^2 - \mathrm{Y}^3 + \cdots) \end{aligned} \tag{20}$$

where

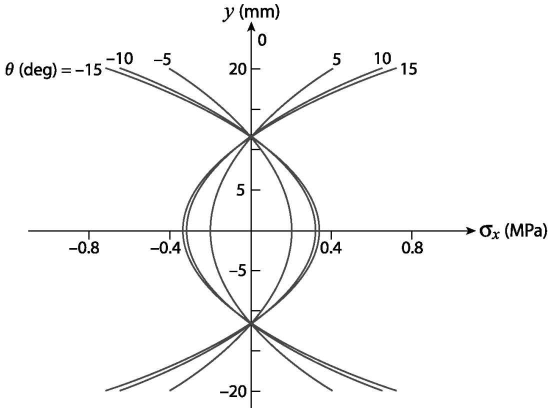

FIG. 2. Normal stresses (σ_x) at free end ($x = 0$ mm) for several grain slope values (θ).

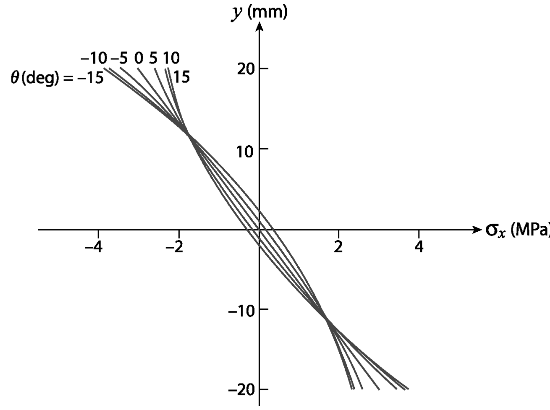

FIG. 3. Normal stresses (σ_x) at fixed end ($x = 80$ mm) for several grain slope values (θ).

$$\mathrm{X} = \frac{2\beta}{\alpha}S_{66} + \left(\frac{\beta}{\alpha}S_{66}\right)^2 \qquad \mathrm{Y} = \frac{\delta}{\gamma}S_{66}$$

For $|\mathrm{Y}| < 1$ as in the present case, the series in Eq. (20) converges very quickly. The first term in Eq. (20) is independent of S_{66}. The second term containing X and Y is a function of S_{66}, although they are not devoid of the other principal compliances.

Equation (19) can now be written

$$\begin{aligned} v_{\substack{x=0\\y=0}} = &\left[\frac{Pl^3}{3I}(m^4S_{11} + 2m^2n^2S_{12} + n^4S_{22})\right.\\ &\left. - \frac{Pb^2l}{12I}\frac{\alpha^2}{\gamma} + \frac{Pb^2l}{2I}m^2n^2(S_{11} + S_{22} - 2S_{12})\right]\\ &+ \left[\frac{Pl^3}{3I}m^2n^2S_{66} - \frac{Pb^2l}{12I}\frac{\alpha^2}{\gamma}\right.\\ &\times (\mathrm{X} - \mathrm{Y})(1 - \mathrm{Y} + \mathrm{Y}^2 - \mathrm{Y}^3 + \cdots)\\ &\left. + \frac{Pb^2l}{8I}(m^2 - n^2)^2S_{66}\right] \end{aligned} \tag{21}$$

The first pair of brackets encloses terms without S_{66}; the second encloses terms with S_{66}.

RESULTS AND DISCUSSION

Mechanical properties for Sitka spruce (Liu 2000) are used for numerical calculations: $E_1 = 11{,}800$ MPa, $E_2 = 2{,}216$ MPa, $G_{12} = 910$ MPa, and $\nu_{12} = 0.37$. The geometrical dimensions and the applied load in Fig. 1 are as follows: $l = 80$ mm, $b = 40$ mm, $h = 10$ mm, and $P = 100$ N. For the grain slope, the assumed values are 0°, ±5°, ±10°, and ±15°.

Figure 2 presents the distribution of the normal stress σ_x at the free end, $x = 0$ mm. For $\theta = 0°$, the stresses are zero as in the case of an isotropic beam. As θ increases from zero, the stresses distribute parabolically from tensile at the upper and lower edges to compressive in the middle. The absolute values of the stresses increase as θ increases but at a decreasing rate. For $\theta > 15°$, the increases become negligible. When θ changes sign, σ_x also changes sign for the same value of y. There are two focal points at $y \approx \pm 12$ mm and $\sigma_x = 0°$, through which all stress curves pass. These stress curves indicate that for $\theta \neq 0°$, the free end is no longer flat because the stresses exist in proportion to the strains.

At the fixed end with $x = 80$ mm, the normal stress σ_x distributions are shown in Fig. 3. For $\theta = 0°$, the stresses fall on a straight line, as in the case of an isotropic beam, with $\sigma_x = \pm 3$ MPa at the upper and lower edges. As θ increases from zero, the stresses form concaved curves toward the first quadrant of the figure, crossing the straight line of $\theta = 0°$ at two focal points at $y \approx \pm 12$ mm and $\sigma_x \approx \pm 1.8$ MPa. At the upper edge of the beam (see Fig. 1), where y is negative and σ_x is positive, the stress increases with θ and reaches its maximum for any specified value of θ; at the lower

TABLE 1. *Deflection of Sitka spruce cantilever.*

$\lvert\theta\rvert$ (degrees)	Equation (19)[a] Flexural (mm)	Equation (19)[a] Shear (mm)	Equation (21)[b] Flexural (mm)	Equation (21)[b] Shear (mm)
0	2.71e-2	3.30e-2	2.71e-2	3.30e-2
5	2.92e-2	3.13e-2	2.70e-2	3.35e-2
10	3.53e-2	2.75e-2	2.69e-2	3.60e-2
15	4.50e-2	2.35e-2	2.71e-2	4.14e-2

[a] Terms without S_{xs} and S_{ss} under flexural; others under shear.
[b] Terms without S_{66} under flexural; others under shear.

edge where y is positive and σ_x is negative, the stress increase reduces its absolute value and reaches its minimum for any specified value of θ. When θ changes from a positive value to a negative value of the same magnitude, the corresponding σ_x and y also change sign but maintain the same magnitude. Thus, for θ = 15°, σ_x has a tensile stress of 3.7 MPa at the upper edge of the beam and a compressive stress of −2.3 MPa at the lower edge; for θ = −15°, it is a tensile stress of 2.3 MPa at the upper edge and a compressive stress of −3.7 MPa at the lower edge. The change from ±3 to ±3.7 MPa is 23%. Note that the stresses at θ = ±15° and ±10° are barely distinguishable. Thus, in this numerical example, the maximum increase in normal stress σ_x due to grain slope θ is 23%.

Results of deflection at the center of the free end are tabulated in Table 1. Since deflection is independent of the sign for θ, absolute values for θ are used in the table. Deflection expressed in terms of the transformed compliances referred to the geometrical axes in Eq. (19) as studied by Kilic et al. (2001) and deflection expressed in terms of the principal compliances referred to the material axes in Eq. (21) are both calculated. Based on Eq. (19), the portion of deflection due to flexural increases with θ, but the portion due to shear decreases with θ. Based on Eq. (21), the portion of deflection due to flexural remains essentially unchanged, but the portion due to shear increases with θ. The results based on Eqs. (19) and (21) are therefore totally incongruous. Since S_{66} in Eq. (21) is an independent material parameter, clearly it should be the one that reflects the effects of shear on deflection in design consideration. As Table 1 indicates, deflection related to S_{66} not only increases with increasing θ, but at an increasing rate. At θ = 0° deflection is 0.033 mm; at θ = 15° it reaches 0.0414 mm, an increase of more than 25%, and it continues to increase.

In lumber grading, each visual stress grade has a very specific maximum permitted grain slope (Hoyle, Jr. 1982). In the design of a wood cantilever, it seems the maximum allowable deflection could be used to limit the maximum permitted grain slope in any specified application, as demonstrated in the numerical example.

CONCLUSIONS

In this study, we analyzed a cantilever of an orthotropic material with a single load at the free end, as shown in Fig. 1. Numerical calculations based on the mechanical properties of Sitka spruce in the longitudinal–radial plane revealed the following:

1. When the beam axis and longitudinal axis coincide, i.e., the grain slope θ is zero, the stress distributions in the beam and the deflection curve of the beam are practically the same as those for an isotropic beam (Timoshenko and Goodier 1951).
2. When the grain slope is zero, the free end of the beam remains flat; when it is different from zero, the free end becomes concave or convex depending on the sign for θ.
3. When the grain slope is zero, the bending stress curve at the fixed end is linear with a positive value at the upper edge and a negative value of equal magnitude at the lower edge; when it is different from zero, the stress curves become nonlinear, crossing the straight line for θ = 0° at two focal points. The stresses at the upper and lower edges may increase or decrease depending on the sign for θ. These changes can be significant, depending on the beam geometry, the material properties, and the applied load.

4. The deflection curve of the beam is independent of the sign for θ. At the center of the free end, deflection increases with increasing θ and at an increasing rate between the considered range of $0° \leq \theta \leq 15°$. The increases are due to the terms containing the principal compliance S_{66}, the inverse of the shear modulus G_{12}. We note that these observations are based on the assumption that E_1 in tension is equal to E_1 in compression.

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