THREE-DIMENSIONAL ANALYSIS OF ELASTIC BEHAVIOR OF WOOD FIBER

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ABSTRACT

An exact solution of stress for each layer of cell wall under tension has been obtained by considering wood fiber as a hollow composite anisotropic circular cylinder. Numerical results of stresses and the relative angle of twist of a single fiber, as well as the values of each layer of the cell wall, are given for six hypothetical wood fibers.

Notation

\( \sigma_{ij} \) Normal stress in \( j \)th layer
\( \tau_{ij} \) Shear stress in \( j \)th layer
\( P \) Tensile force at ends
\( a_j \) Radius of fiber at the outer edge of \( j \)th layer
\( A, B, C, D, E, F \) Unknown constants
\( \alpha_{ma}(j) \) Compliances of \( j \)th layer
\( \beta_{ma}(j) \) Coefficients of deformation of \( j \)th layer
\( u_{ij} \) Displacement in \( j \)th layer
\( E_i \) Young's modulus
\( G_{ij} \) Moduli of rigidity
\( 
\) Directional Poisson's ratios
\( T \) Torsional rigidity of a single fiber
\( T^{(j)} \) Torsional rigidity of \( j \)th layer
\( D_{total} \) Total relative angle of twist of a single fiber
\( \Phi, \Psi \) Stress functions

INTRODUCTION

In recent years, considerable effort has been expended in the development of fiber mechanics, and as a consequence, the mechanical behavior of the cell wall has become very important. The problems in general may be classified as follows: (1) determination of the elastic constants of crystalline cellulose on the microscopic level; (2) determination of the elastic constants of the cell wall on the microscopic level; and (3) stress distribution on each layer of the cell wall when the fiber is deformed.

The first problem has been extensively studied by many authors (Gillis 1969; Gillis, Mark, and Tang 1969; Jaswon, Gillis, and Mark 1968; Lyons 1958; Mark et al. 1969; Meredith 1946; Sakusada, Nukushina, and Ito 1962; Treloar 1960). In dealing with the second problem, several authors have used the concept of layered system (Cave 1969; 1969; Gillis 1970; Mark 1967; Mark and Gillis 1970; Schniewind 1969; Schniewind and Barrett 1969). Literature in the third category, however, is limited. As is known, the theoretical study of stress in each layer of the cell wall is restricted in two-dimensional analysis by cutting an element from the wall of tubular wood fiber (Mark 1967; Mark and Gillis 1970; Schniewind 1969; Schniewind and Barrett 1969).

It is quite obvious that a significant change of stress in each layer of the wall will arise when considered as a three-dimensional system. This has been demonstrated by the experimental testing of a helically reinforced plastics pipe (Mark 1967). The rotation test of a single fiber (pine) under tension has been reported by Mark and Gillis (1970). No literature on the theoretical analysis of twisting of a single fiber has been found by the author.

In this paper, the stresses and the relative angle of twist on each layer of cell wall are
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from a material with cylindrical anisotropy, and subjected to a tensile force $P$ at the ends (Fig. 1). We assume that: (1) the axis of anisotropy coincides with the geometric axis of the cylinder, and (2) the stresses that act on the end surfaces reduce to forces and to twisting moments directed along the axis.

According to Lekhnitskii (1963), the governing partial differential equations of a hollow cylinder with body forces absent take the forms:

$$L_1' + L_2' = 0,$$
$$L_1'' + L_2' = \frac{F}{r} - 2D \quad (1)$$

Here $F$ is an arbitrary constant, $a_{st}$ is an elastic constant, $D$ is the relative angle of twist, and $L_1', L_2', L_2''$ are differential operators defined as follows:

$$L_1' = \beta_{14} \frac{\partial^2}{\partial r^2} + \beta_{24} \frac{1}{r^2} \frac{\partial}{\partial r} + \beta_1 \frac{1}{r} \frac{\partial}{\partial r} + \beta_2 \frac{1}{r} \frac{\partial}{\partial \theta},$$
$$L_2' = \frac{1}{r} \left( \beta_{14} - 2\beta_{24} \right) \frac{\partial}{\partial r}^2,$$
$$L_2'' = -\beta_{24} \frac{\partial^2}{\partial r^2} - \beta_{14} \frac{1}{r^2} \frac{\partial}{\partial r} + \beta_2 \frac{1}{r} \frac{\partial}{\partial \theta},$$
$$L_3' = -\beta_{14} \frac{3}{r^3} + \beta_{24} \frac{1}{r^2} \frac{\partial}{\partial r} + \beta_2 \frac{1}{r} \frac{\partial}{\partial \theta},$$
$$L_4' = \beta_{24} \frac{3}{r^2} + \left( 2\beta_{14} + \beta_{24} \right) \frac{1}{r^2} \frac{\partial^4}{\partial r^2 \partial \theta^2} + \beta_{14} \frac{1}{r^2} \frac{\partial^2}{\partial r \partial \theta^2},$$
$$L_4'' = 2\beta_{14} \frac{1}{r^2} \frac{\partial^2}{\partial r \partial \theta^2} - \beta_{14} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \beta_{24} \frac{1}{r^2} \frac{\partial^2}{\partial r \partial \theta^2} + \beta_{24} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

where $\beta_{mn}$'s are the elastic coefficients of deformations.

**THE BASIC EQUATIONS AND SOLUTION**

Let us consider wood fiber as a composite anisotropic cylinder of finite length, made calculated by considering the wood fiber as a hollow composite anisotropic circular cylinder subjected to tension. It is desirable to model the fiber as a cylinder of rectangular or a square cross section, possessing two different sets of elastic constants for the tangential and radial walls. To obtain more precise results, it would be desirable to round corners in a manner to simulate the geometry of a fiber. The analysis of such a cylinder is very complex, however. Consequently, in this work the fiber was assumed to be of circular cross section. Such an approach will yield results more nearly consistent with the actual behavior of the fiber than the two-dimensional analysis reported to date.
Because the stress distribution in a hollow cylinder is symmetrical about the center axis, the stress functions $\Phi$ and $\Psi$ will depend only on the radius $r$ (Tang and Adams 1970). Assuming plane deformation of a composite cylinder (Fig. 1), then equations (1) are satisfied if we take the stress components of the $j$th layer in the following form:

\[
\sigma_r^{(j)} = \frac{\sigma_r^{(j)}(r)}{r} = E^{(j)} \frac{\sigma_{\theta }^{(j)}}{\beta_{14}^{(j)}} r^{-1} + A^{(j)} \\
+ B^{(j)} r^{-k_j} - C^{(j)} r^{-k_j-1} + D^{(j)} G^{(j)} r, \tag{3}
\]

\[
\sigma_\theta^{(j)} = -\sigma_r^{(j)} \frac{\sigma_{\theta }^{(j)}}{\beta_{14}^{(j)}} r^{-1} + A^{(j)} G^{(j)} - C^{(j)} k_j r^{-k_j-1} + 2B^{(j)} G^{(j)} r, \tag{4}
\]

\[
\sigma_{z}^{(j)} = \Psi^{(j)} r^{(j)} = -E^{(j)} F^{(j)} \frac{\sigma_{\phi}^{(j)}}{\beta_{14}^{(j)}} \\
- A^{(j)} G^{(j)} - B^{(j)} g^{(j)} r^{-k_j} - C^{(j)} k_j r^{-k_j-1} + D^{(j)} H^{(j)} r, \tag{5}
\]

\[
\sigma_{r\theta}^{(j)} = \sigma_{\theta z}^{(j)} = \sigma_{rz}^{(j)} = 0 \tag{7}
\]

Here all the superscripts $(j)$ or the subscripts $(j)$ indicate particular layers. $A^{(j)}$, $B^{(j)}$, $C^{(j)}$, $D^{(j)}$, $E^{(j)}$, $F^{(j)}$ are the arbitrary constants to be determined from boundary conditions, and $D^{(j)}$ is the constant equal to the relative angle of twist. The constants $A_{mn}$ are the elastic coefficients of the material. Other constants are as follows:

\[
\beta_{mn}^{(j)} = \left[ a_{mn}^{(j)} - \frac{d_{m3}^{(j)} a_{33}^{(j)}}{a_{33}^{(j)}} \right]_{m,n=1,2,4,5} \tag{a}
\]

\[
k_j = \left[ \frac{\beta_{11}^{(j)} \beta_{44}^{(j)} - \beta_{14}^{(j)} \beta_{41}^{(j)}}{\beta_{22}^{(j)} \beta_{44}^{(j)} - \beta_{24}^{(j)} \beta_{42}^{(j)}} \right]^{1/2} \tag{b}
\]

\[
g_{2k}^{(j)} = \left[ \frac{\beta_{44}^{(j)} + k_2^{(j)}}{\beta_{44}^{(j)}} \right]^{1/2} \tag{c}
\]

\[
g_{2k}^{(j)} = \left[ \frac{\beta_{11}^{(j)} - a_{22}^{(j)}}{\beta_{22}^{(j)} - \beta_{24}^{(j)}} \right] \tag{d}
\]

\[
H^{(j)} = \left[ \frac{\beta_{11}^{(j)} - a_{22}^{(j)}}{\beta_{22}^{(j)} - \beta_{24}^{(j)}} \right] \tag{e}
\]

It follows from equations (3-7) that the displacements in $j$th layer can be expressed as

\[
u_r^{(j)} = \int \left( a_{n1}^{(j)} \sigma_r^{(j)} + d_{11}^{(j)} \sigma_r^{(j)} + a_{n2}^{(j)} \sigma_\theta^{(j)} + d_{12}^{(j)} \sigma_\theta^{(j)} + a_{n3}^{(j)} \sigma_z^{(j)} + d_{13}^{(j)} \sigma_z^{(j)} \right) dr, \tag{9}
\]

The boundary conditions on the cylindrical surfaces are as follows:

\[
\sigma_r^{(1)} = 0 \quad \text{at} \quad r = a_e \tag{10}
\]

\[
\sigma_r^{(4)} = 0 \quad \text{at} \quad r = a_i \tag{11}
\]

At the contact surfaces of adjacent layers, the following stress and displacement relations must hold:

\[
\sigma_r^{(j)} = \sigma_{r}^{(j+1)} \tag{12}
\]

\[
u_r^{(j)} = \nu_r^{(j+1)} \tag{13}
\]

\[
\sigma_{\theta z}^{(j)} = \sigma_{\theta z}^{(j+1)} \tag{14}
\]

The boundary conditions for $r = a_j$ $(j = 1, 2, 3)$ at the ends are

\[
\sum_{j=1}^{3} \frac{a_i^{(j)} \sigma_{z}^{(j)} r^d r = \frac{D}{2\pi} , \quad \text{and} \tag{15}
\]

\[
\sum_{j=1}^{3} \frac{a_i^{(j)} \sigma_{z}^{(j)} r^d r = 0 . \tag{16}
\]
In determining displacements and requiring that they be a single-valued function of the coordinates, it is necessary that:

\[ E^{(j)} = 0 \]
and

\[ A^{(j)} = F^{(j)} h^{(j)} \]

where

\[ h^{(j)} = \left[ \begin{array}{c} a_{11} - a_{21} \beta_{14} - a_{14} (\beta_{14} - \beta_{21}) \\ a_{12} - a_{22} \beta_{24} - a_{24} (\beta_{24} - \beta_{12}) \end{array} \right] \].

It should be noted also that:

\[ F^{(j)} = F^{(j+1)} = F \]

For the generalized plane deformation problems, the axial strain is constant.

\[ \frac{\partial \omega}{\partial z} = \text{constant}. \]

Consequently, when a composite cylinder is subjected to a tensile force \( P \) at the ends, the axial strain of all layers must be identical; hence:

\[ \frac{\partial \omega^{(j)}}{\partial z} = \frac{\partial \omega^{(j+1)}}{\partial z} = F. \]

Using the boundary conditions given in (10), we obtain:

\[ F h^{(j)} + B a_{o}^{(j)} + C a_{4}^{(j)} k_{j}^{-1} + D a_{4}^{(j+1)} = 0, \quad j = \text{constant}. \]

\[ F h^{(j+1)} + C a_{4}^{(j+1)} k_{j}^{-1} = 0, \quad j = \text{constant}. \]

Furthermore, the boundary conditions of (11) give

\[ F (h^{(j+1)} - h^{(j)}) + B (a_{j}^{(j+1)} - a_{j}^{(j)}) k_{j}^{-1} + C (a_{j}^{(j+1)} - a_{j}^{(j)}) k_{j}^{-1} + D (a_{j}^{(j+1)} - a_{j}^{(j+1)}) k_{j}^{-1} \]

\[ = 0, \quad j = 1, 2, 3. \]

\[ F (M_{1}^{(j)} - M_{1}^{(j)}) a_{j} + B (a_{j}^{(j+1)} - a_{j}^{(j)}) k_{j}^{-1} \]

\[ + C (a_{j}^{(j+1)} - a_{j}^{(j)}) k_{j}^{-1} = 0, \quad j = 1, 2, 3. \]

\[ \beta_{14}^{(j)} a_{11}^{(j)} - \beta_{14}^{(j+1)} a_{14}^{(j+1)} k_{j}^{-1} + C \beta_{14}^{(j+1)} k_{j}^{-1} \]

\[ = 0, \quad j = 1, 2, 3. \]

where \( M_{1}^{(j)} = (\beta_{14}^{(j)} - \beta_{14}^{(j+1)} - a_{11}^{(j)} - a_{14}^{(j+1)} a_{14}^{(j+1)} k_{j}^{-1}) \)

\[ M_{2}^{(j)} = \beta_{14}^{(j)} a_{11}^{(j)} - \beta_{14}^{(j+1)} a_{14}^{(j+1)} k_{j}^{-1} \]

\[ M_{3}^{(j)} = \beta_{14}^{(j)} a_{11}^{(j)} - \beta_{14}^{(j+1)} a_{14}^{(j+1)} k_{j}^{-1} \]

\[ M_{4}^{(j)} = (\beta_{14}^{(j)} a_{11}^{(j)} + a_{14}^{(j+1)} a_{14}^{(j+1)} k_{j}^{-1}) \]

and

\[ \frac{\partial^{(j)}}{\partial z} - \frac{\partial^{(j+1)}}{\partial z} + h^{(j)} = 0, \quad j = 1, 2, 3. \]

From the boundary conditions given in (12), we have:

\[ \sum_{j=1}^{4} \left\{ \frac{E}{\beta_{14}^{(j)}} a_{11}^{(j)} - h^{(j)} a_{14}^{(j)} \left( a_{14}^{(j)} - a_{14}^{(j+1)} \right) \right\} = 0. \]
These unknown constants $B^{(j)}$, $C^{(j)}$, $D^{(j)}$, $(j = 1, 2, 3, 4)$, and $F$ can be determined from equations (14)-(20). Consequently, all the data necessary for calculation of stresses and displacements are available.

**NUMERICAL EXAMPLES**

It is known that the microfibrillar angles in $S_2$ and $S_3$ layers on the tangential wall of wood fiber are different from those on the radial wall except in the $S_1$ layer and $M + P$ layer (middle lamella and primary wall). The variation of these microfibrillar angles and the area percentage of the various cell-wall layers in the fiber are listed in Table 1 (Mark and Gillis 1970).

To simplify the numerical calculations, models were used with structure configurations as listed in Table 2. Assuming that the exterior radius of each model is a unity, then, according to Table 2, the corresponding radius of each layer is found to be:

- $a_s$ (to the inner edge of $S_1$): 0.82566
- $a_t$ (to the inner edge of $S_2$): 0.84700
- $a_r$ (to the inner edge of $S_3$): 0.95643
- $a_m$ (to the inner edge of $M + P$): 0.98396
- $a_o$ (to the outer edge of $M + P$): 1.00000

The elastic constants of each layer were calculated by using an approach similar to that of Gillis (1970). Three cases are considered. The elastic constants of crystalline cellulose are listed in Table 3.

**Table 1. Microfibril angles and area percentages of the various cell-wall layers in the fiber**

<table>
<thead>
<tr>
<th>Layer</th>
<th>Filament winding angles</th>
<th>Area % of the various cell-wall layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial wall</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M + P$</td>
<td>90°</td>
<td>7.40</td>
</tr>
<tr>
<td>$S_1$</td>
<td>± 80°</td>
<td>11.84</td>
</tr>
<tr>
<td>$S_2$</td>
<td>36°</td>
<td>45.88</td>
</tr>
<tr>
<td>$S_3$</td>
<td>64°</td>
<td>8.88</td>
</tr>
<tr>
<td>Tangential wall</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M + P$</td>
<td>90°</td>
<td>2.60</td>
</tr>
<tr>
<td>$S_1$</td>
<td>± 80°</td>
<td>4.94</td>
</tr>
<tr>
<td>$S_2$</td>
<td>20°</td>
<td>16.12</td>
</tr>
<tr>
<td>$S_3$</td>
<td>30°</td>
<td>2.34</td>
</tr>
</tbody>
</table>

**Table 2. Data for model fiber**

<table>
<thead>
<tr>
<th>Model</th>
<th>Layer</th>
<th>Filament winding angles</th>
<th>Area % of the various cell-wall layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>I $M + P$</td>
<td>90°</td>
<td>10.00</td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>± 80°</td>
<td>16.78</td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>36°</td>
<td>62.00</td>
<td></td>
</tr>
<tr>
<td>$S_3$</td>
<td>64°</td>
<td>11.22</td>
<td></td>
</tr>
<tr>
<td>II $M + P$</td>
<td>90°</td>
<td>10.00</td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>± 80°</td>
<td>16.78</td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>20°</td>
<td>62.00</td>
<td></td>
</tr>
<tr>
<td>$S_3$</td>
<td>30°</td>
<td>11.22</td>
<td></td>
</tr>
</tbody>
</table>

The values used to describe the associated matrix material and the proportions of matrix and framework in each layer are as given by Mark and Gillis (1970), viz.

- $E = 0.200 \times 10^{11}$ dynes/cm²
- $G = 0.0769 \times 10^{11}$ dynes/cm²
- $\mu = 0.30$

and Matrix (%) Framework (%)

<table>
<thead>
<tr>
<th>Layer</th>
<th>M + P</th>
<th>S1, S2 and S3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M + P$</td>
<td>89.9</td>
<td>46.9</td>
</tr>
<tr>
<td>S1, S2 and S3</td>
<td>10.1</td>
<td>53.1</td>
</tr>
</tbody>
</table>

**Table 3. Elastic constants of crystalline cellulose in the cell wall (E and G in units of $10^{11}$ dyn/cm²)**

<table>
<thead>
<tr>
<th>Case</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
<th>$C_{23}$</th>
<th>$\rho_{12}$</th>
<th>$\rho_{13}$</th>
<th>$\rho_{23}$</th>
<th>$\mu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.13</td>
<td>5.65</td>
<td>1.68</td>
<td>0.049</td>
<td>0.236</td>
<td>0.018</td>
<td>-0.0016</td>
<td>0.041</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.49</td>
<td>31.90</td>
<td>3.73</td>
<td>0.023</td>
<td>0.323</td>
<td>0.039</td>
<td>0.0002</td>
<td>0.041</td>
<td>3.5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.72</td>
<td>13.40</td>
<td>2.72</td>
<td>0.440</td>
<td>0.660</td>
<td>0.440</td>
<td>0.100</td>
<td>0.100</td>
<td>11†</td>
<td></td>
</tr>
</tbody>
</table>

*The 1-direction corresponds to radial axis, the 2-direction corresponds to the filament length, and the 3-direction completes a right-handed orthogonal set (see Fig. 2).†It is assumed that Case 3 is transverse-isotropic, because only the coefficients $E_{ij}$, $C_{ij}$, $\mu_{ij}$ and $\mu_{ij}$ are available.
TABLE 4. Elastic constants of the layers of the cell wall (E and G in units of $10^{11}$ dynes/cm$^2$)

<table>
<thead>
<tr>
<th>Case</th>
<th>Layer</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\mu_{12}$</th>
<th>$\mu_{13}$</th>
<th>$\mu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>M + P</td>
<td>0.22765</td>
<td>0.59777</td>
<td>0.2578</td>
<td>0.07415</td>
<td>0.09707</td>
<td>0.10079</td>
<td>0.18853</td>
<td>0.20425</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>S, S, S</td>
<td>0.83588</td>
<td>3.09170</td>
<td>0.36245</td>
<td>0.06137</td>
<td>0.15161</td>
<td>0.04208</td>
<td>0.27898</td>
<td>0.11425</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>M + P</td>
<td>0.26215</td>
<td>3.24450</td>
<td>0.23567</td>
<td>0.10001</td>
<td>0.07145</td>
<td>0.07311</td>
<td>0.27527</td>
<td>0.02380</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>S, S, S</td>
<td>1.41660</td>
<td>17.03000</td>
<td>0.72241</td>
<td>0.04535</td>
<td>0.1847</td>
<td>0.05339</td>
<td>0.15812</td>
<td>0.10305</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>M + P</td>
<td>0.44601</td>
<td>1.5295</td>
<td>0.37308</td>
<td>0.10980</td>
<td>0.12651</td>
<td>0.10980</td>
<td>0.09570</td>
<td>0.20499</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>S, S, S</td>
<td>0.39908</td>
<td>7.44880</td>
<td>0.39370</td>
<td>0.17621</td>
<td>0.20013</td>
<td>0.17621</td>
<td>0.02590</td>
<td>0.20581</td>
<td></td>
</tr>
</tbody>
</table>

The elastic constants of each of the layers with respect to their principal axes (Fig. 2) are listed in Table 4. The elastic compliances of the various layers with respect to the geometric axes (see Fig. 1) can be easily obtained by using tensor transformations. The numerical calculation is straightforward and hence is not given here. With these elastic compliances and radius of each layer known, the stresses were calculated by using equations (3)-(6). The distributions of all the stresses in the radial direction of the cell wall are plotted in Figs. 3-6, and the relative angle of twist-
ing of each layer is shown also. According to Lekhnitskii (1963), the total torsional rigidity \((T)\) of a composite anisotropic cylinder equals the sum of the torsional rigidities of each layer \((T^{(i)})\); that is:

\[
T = \sum_{j=1}^{n} T^{(j)},
\]

where \(n\) is the total number of layers in the cylinder. Hence, the total relative angle of twist of a single fiber can be expressed as

\[
D_{\text{total}} = \frac{\sum_{j=1}^{4} T^{(j)} D^{(j)}}{\sum_{j=1}^{4} T^{(j)}}.
\]

The results of all cases are tabulated in Table 5.

Table 5. *Theoretical values of the relative angle of twist of two model fibers*

<table>
<thead>
<tr>
<th>Model</th>
<th>Case</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(S_4)</th>
<th>(M+P)</th>
<th>Single fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>645.94</td>
<td>63.13</td>
<td>147.67</td>
<td>329.26</td>
<td>110.90</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>787.47</td>
<td>57.02</td>
<td>186.06</td>
<td>122.77</td>
<td>103.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>567.52</td>
<td>357.65</td>
<td>88.81</td>
<td>228.05</td>
<td>334.78</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>150.53</td>
<td>158.30</td>
<td>191.99</td>
<td>428.08</td>
<td>193.17</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>12.92</td>
<td>31.83</td>
<td>106.85</td>
<td>70.50</td>
<td>41.79</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>252.81</td>
<td>59.97</td>
<td>57.97</td>
<td>148.86</td>
<td>89.14</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 5. Variation of $\sigma_{zz}$ along $r$ direction.

Fig. 6. Variation of $\sigma_{eq}$ along $r$ direction.
radial stress cannot be obtained
layers.

Applying the law of mixture with the values
given in Table 5, the values of the relative
twisting angle of a single fiber with area
percentages of tangential wall and radial
wall as shown in Table 1 were calculated.
The angles of relative twisting were as fol-

Case 1: 121.24°
Case 2: 95.72°
Case 3: 283.60°

These calculations contrast with the experi-
mental data reported by Mark and Gillis
(1970) of 317.70°; an average of six tests.

To illustrate the difference between two-
and three-dimensional analyses, the ratios
of longitudinal stress $\sigma_{l}$, tangential stress
$\sigma_{t}$, radial stress $\sigma_{r}$, and the shear stress $\sigma_{l,r}$
to the externally applied tensile stress $\sigma$
in the fiber direction for cases I3 and II3,
along with values from two-dimensional
analysis reported by Mark and Gillis (1970),
are listed in Table 6. It is evident from
this comparison that the three-dimensional
models employed here convey a much dif-
ferent impression of the state of stress in
the cell wall than two-dimensional models
reported in the literature to date.

CONCLUSIONS

The maximum radial stress for cases II,
I2, and I1 occurred at the boundary of the
$S_2$ and $S_1$ layers, while in cases I3, II2, and
II3 it is at the boundary of the $S_3$ and $S_2$
layers. It should be noted here that the
radial stress cannot be obtained from a two-
dimensional analysis. In most cases, the
tangential stresses are at a maximum in the
$S_1$ layer except in cases I3 and II3, where it
occurs in the $S_1$ layer. Previous studies of
two-dimensional analyses of the elastic be-
havior of the wood fiber have led to the
conclusion that the tangential stress in the
cell wall is always at a maximum in the $S_1$
layer.

The points of zero shear stress in all cases
occurred either inside the $S_2$ layer or inside
the $S_3$ layer. All previous two-dimensional
analyses of the stress on the cell wall re-
ported that shear stress does not exist in the
$M + P$ layer. This is in contrast to the
three-dimensional analysis. In addition, it
is known that shear stress exists in all radial
directions of an anisotropic tube when it is
subjected to a tensile force (Lekhnitskii,
1963).

These results suggest that a two-dimen-
sional analysis is neither sufficient nor de-
pendable for the study of elastic models
of the cell-wall. Two-dimensional analysis
supplies no information about the relative
twisting angle of a single fiber and each
layer in the cell wall, whereas three-dimen-
sional analysis does. It can be seen from
Table 5 that in case 1, model II gives the
upper bound of the relative twisting angle
of a single fiber, while model I gives the
lower bound. This situation is reversed in
cases 2 and 3. It is particularly interesting
to point out that the experimental value of
the relative twisting angle of a single fiber

TABLE 6. Comparison of the results of stresses from two- and three-dimensional analyses

<table>
<thead>
<tr>
<th>Three-dimensional analysis</th>
<th>Mark and Gillis (1970)</th>
<th>Two-dimensional analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{l}$</td>
<td>$\sigma_{t}$</td>
</tr>
<tr>
<td>Max</td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td>M + P</td>
<td>-0.940 -0.945</td>
<td>2.673 2.636</td>
</tr>
<tr>
<td>$S_1$</td>
<td>-0.907 -0.908</td>
<td>4.492 4.344</td>
</tr>
<tr>
<td>$S_2$</td>
<td>-1.021 -1.936</td>
<td>-1.657 -1.360</td>
</tr>
<tr>
<td>$S_3$</td>
<td>-0.767 -0.819</td>
<td>1.712 1.436</td>
</tr>
<tr>
<td>Radial wall</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M + P</td>
<td>-0.864 -0.860</td>
<td>1.964 1.990</td>
</tr>
<tr>
<td>$S_1$</td>
<td>-0.844 -0.846</td>
<td>3.071 2.970</td>
</tr>
<tr>
<td>$S_2$</td>
<td>-1.186 -1.311</td>
<td>-0.584 -0.759</td>
</tr>
<tr>
<td>$S_3$</td>
<td>-1.891 -1.915</td>
<td>-1.927 -1.984</td>
</tr>
</tbody>
</table>
obtained by Mark and Gillis (1970) falls within the bounds of case 3, in which the elastic constants of cellulose and matrix given by them have been used. Hence, one can conclude that three-dimensional analysis offers a more complete and rigorous solution of the elastic behavior of the wood fiber.

REFERENCES


